

BOUNDED GEOMETRY AND LEAVES

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ABSTRACT. The main theorem states that any complete connected Riemannian manifold of bounded geometry can be isometrically realized as a leaf with trivial holonomy in a compact Riemannian foliated space.

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1. INTRODUCTION

Recall that a foliated space $X \equiv (X, \mathcal{F})$ of dimension n is a topological space X equipped with a partition \mathcal{F} into connected manifolds (leaves) so that X can be locally described as a product $B \times Z$, where B is an open ball in \mathbb{R}^n and Z any topological space (local transversal), and the slices $B \times \{*\}$ correspond to open sets in the leaves. This \mathcal{F} is called a foliated structure or lamination. Foliated spaces are usually assumed to be Polish to get better properties. Many basic notions about foliations can be obviously extended to foliated spaces, like foliated charts, plaques, foliated atlas, holonomy pseudogroup, holonomy group and holonomy covering of the leaves, minimality, transitivity, foliated maps, etc. Some basic results can be extended as well; for instance, there is an obvious version of the Reeb local stability theorem, and the union of leaves without holonomy is a meager subset if X is second countable. Interesting classes of foliated spaces show up in several areas of mathematics, like in dynamics, arithmetics, tessellations, graphs and foliation theory (minimal sets).

A C^∞ foliated structure is given by a foliated atlas whose changes of coordinates are leafwise C^∞ , with ambient-space-continuous leafwise derivatives of arbitrary order. This gives rise to the concept of C^∞ foliated space. To emphasize the difference, the foliated structure underlying a C^∞ foliated structure may be called topological. On a C^∞ foliated space $X \equiv (X, \mathcal{F})$, the concept of C^∞ function is defined by requiring that its local expressions, using foliated coordinates, are leafwise C^∞ , with ambient-space-continuous leafwise partial derivatives of arbitrary order. C^∞ bundles and sections also make sense on X , defined by requiring that their local descriptions are given by C^∞ functions in the above sense. For instance, the tangent bundle TX (or $T\mathcal{F}$) is the C^∞ vector bundle on X that consists of the vectors tangent to the leaves, and a Riemannian metric on X consists of Riemannian metrics on the leaves fitting together nicely to form a C^∞ section on X . This gives rise to the concept of Riemannian foliated space.

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C^∞ foliated maps between C^∞ foliated spaces can be similarly defined; in particular, C^∞ foliated immersions, submersions, (local) diffeomorphisms and (local) embeddings between C^∞ foliated spaces have obvious meanings. If a homeomorphism between C^∞ foliated spaces is C^∞ and its restrictions to the leaves are diffeomorphisms, then it is a C^∞ diffeomorphism, as follows easily from the continuity of the inversion of C^∞ diffeomorphisms between C^∞ manifolds with respect to the C^∞ topology [18, p. 64, Exercise 9]. Several results about foliated spaces have obvious C^∞ versions, like the Reeb local stability theorem.

Standard references about foliated spaces are [23], [4, Chapter 11], [5, Part 1] and [13]. See also [1, Section 2.1] for a quick summary of what is needed here.

On the other hand, recall that a Riemannian manifold M is said to be of bounded geometry when it has a positive injectivity radius, and the m -th covariant derivative of the curvature tensor has uniformly bounded norm for all order m ; in particular, M is complete by the positivity of the injectivity radius. The following are typical examples where bounded geometry holds: coverings of closed connected Riemannian manifolds, connected Lie groups with left invariant metrics, and leaves of compact Riemannian foliated spaces. More examples can be produced by using compactly supported perturbations of given Riemannian manifolds of bounded geometry. In fact, any smooth manifold admits a metric of bounded geometry [14]. We will focus in the case of leaves of compact Riemannian foliated spaces, showing that this example indeed characterizes bounded geometry.

Theorem 1.1. *Any connected Riemannian manifold of bounded geometry is isometric to a leaf with trivial holonomy of some compact Riemannian foliated space.*

It is commonly accepted that such a result should be true, and that it should follow by using the closure of the canonical embedding of the manifold into the Gromov space \mathcal{M}_* of pointed proper metric spaces [15], [16, Chapter 3], or, better, into its smooth version, the space $\mathcal{M}_*^\infty(n)$ of isometry classes of pointed complete connected Riemannian n -manifolds with the topology defined by the C^∞ convergence [24, Chapter 10, Section 3.2], [1, Theorem 1.2]. However, to the authors knowledge, no complete proof has been given so far.

A complete connected Riemannian n -manifold M is called non-periodic (respectively, locally non-periodic) if $\text{Iso}(M) = \{\text{id}_M\}$ (respectively, the canonical projection $M \rightarrow \text{Iso}(M) \backslash M$ is a covering map), where $\text{Iso}(M)$ denotes the isometry group of M . The non-periodic and locally non-periodic manifolds define subspaces of $\mathcal{M}_*^\infty(n)$ respectively denoted by $\mathcal{M}_{*,\text{np}}^\infty(n)$ and $\mathcal{M}_{*,\text{lnp}}^\infty(n)$. There is a canonical map $\iota_M : M \rightarrow \mathcal{M}_*^\infty(n)$, given by $\iota_M(x) = [M, x]$ (the isometry class of (M, x)), which induces a continuous injection $\bar{\iota}_M : \text{Iso}(M) \backslash M \rightarrow \mathcal{M}_*^\infty(n)$. The images of all possible maps ι_M form a partition $\mathcal{F}_*(n)$ of $\mathcal{M}_*^\infty(n)$. The restriction of $\mathcal{F}_*(n)$ to $\mathcal{M}_{*,\text{lnp}}^\infty(n)$ is denoted by $\mathcal{F}_{*,\text{lnp}}(n)$. For $n \geq 2$, $\mathcal{M}_{*,\text{lnp}}^\infty(n)$ is open and dense in $\mathcal{M}_*^\infty(n)$, and $\mathcal{F}_{*,\text{lnp}}(n)$ is a Riemannian foliated space of dimension n so that each map $\iota_M : M \rightarrow \text{im } \iota_M$ is a local isometry and the holonomy covering of the leaf $\text{im } \iota_M$ [1, Theorem 1.3]; in particular, $\mathcal{M}_{*,\text{np}}^\infty(n)$ is the union of leaves with trivial holonomy. Moreover $\text{Cl}_\infty(\text{im } \iota_M)$ is compact if and only if M is of bounded geometry [1, Theorem 12.3] (see also [8], [24, Chapter 10, Sections 3 and 4]), where Cl_∞ denotes the closure operator in $\mathcal{M}_*^\infty(n)$. Then, analyzing the cases where $\text{Cl}_\infty(\text{im } \iota_M) \subset \mathcal{M}_{*,\text{lnp}}^\infty(n)$, a version of Theorem 1.1 follows assuming restrictions on M [1, Theorem 1.5].

To prove Theorem 1.1 with complete generality, we refine the above arguments as follows. Fix a separable Hilbert space \mathbb{E} and any natural¹ n . Consider pairs (M, f) and triples (M, f, x) , where M is a complete connected Riemannian n -manifold, $f \in C^\infty(M, \mathbb{E})$ and $x \in M$. An *equivalence* $\phi : (M, f) \rightarrow (N, h)$ is an isometry $\phi : M \rightarrow N$ such that $\phi^*h = f$. If moreover distinguished points, $x \in M$ and $y \in N$, are preserved, then $\phi : (M, f, x) \rightarrow (N, h, y)$ is called a *pointed equivalence*. The group of self equivalences of (M, f) is denoted by $\text{Iso}(M, f)$. If there is a pointed equivalence $(M, f, x) \rightarrow (N, h, y)$, then the triples (M, f, x) and (N, h, y) are declared to be *equivalent*. The equivalence class of each (M, f, x) is denoted by $[M, f, x]$. Let $\widehat{\mathcal{M}}_*(n)$ denote the set² of such equivalence classes.

Definition 1.2. For each $m \in \mathbb{N}$, a sequence $[M_i, f_i, x_i]$ in $\widehat{\mathcal{M}}_*(n)$ is said to be C^m convergent to $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$ if, for each compact domain³ $\Omega \subset M$ containing x , there is a pointed C^{m+1} embedding $\phi_i : (\Omega, x) \rightarrow$

¹It is assumed that 0 is natural.

²Like in the cases of \mathcal{M}_* and $\mathcal{M}_*^\infty(n)$, without loss of generality, it can be assumed that the underlying set of any such M is contained in \mathbb{R} , so that $\widehat{\mathcal{M}}_*(n)$ becomes a well defined set.

³Here, a *domain* in M is a connected C^∞ submanifold, possibly with boundary, of the same dimension as M .

(M_i, x_i) for each large enough i such that $\phi_i^* g_i \rightarrow g|_\Omega$ and $\phi_i^* f_i \rightarrow f|_\Omega$ as $i \rightarrow \infty$ with respect to the C^m topology [18, Chapter 2]. If $[M_i, f_i, x_i]$ is C^m convergent to $[M, f, x]$ for all m , then it is said that $[M_i, f_i, x_i]$ is C^∞ convergent to $[M, f, x]$.

It is not completely obvious that this C^∞ convergence satisfies the conditions to define a topology [21], [17]. Thus the following result is not trivial.

Theorem 1.3. *The C^∞ convergence in $\widehat{\mathcal{M}}_*(n)$ describes a Polish topology.*

The topology given by Theorem 1.3 will be called the C^∞ topology, and the corresponding space is denoted by $\widehat{\mathcal{M}}_*^\infty(n)$. The closure operator in this space will be denoted by $\widehat{\text{Cl}}_\infty$. The following maps are canonical and continuous: a *forgetful* map $\widehat{\mathcal{M}}_*^\infty(n) \rightarrow \mathcal{M}_*^\infty(n)$, $[M, f, x] \mapsto [M, x]$, and an *evaluation* map $\text{ev} : \widehat{\mathcal{M}}_*^\infty(n) \rightarrow \mathbb{E}$, $[M, f, x] \mapsto f(x)$. Note that $\text{ev} : \widehat{\mathcal{M}}_*(0) \rightarrow \mathbb{E}$ is a homeomorphism. Moreover, for each complete connected Riemannian n -manifold M and any $f \in C^\infty(M, \mathbb{E})$, there is a canonical continuous map $\hat{\iota}_{M,f} : M \rightarrow \widehat{\mathcal{M}}_*^\infty(n)$, given by $\hat{\iota}_{M,f}(x) = [M, f, x]$, which induces a continuous injection $\bar{\iota}_{M,f} : \text{Iso}(M, f) \setminus M \rightarrow \mathcal{M}_*^\infty(n)$. The images of the maps $\hat{\iota}_{M,f}$ form a natural partition of $\widehat{\mathcal{M}}_*^\infty(n)$, denoted by $\widehat{\mathcal{F}}_*(n)$. Let $C_{\text{imm}}^\infty(M, \mathbb{E})$ be the set of C^∞ immersions $M \rightarrow \mathbb{E}$, and let $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ be the $\widehat{\mathcal{F}}_*(n)$ -saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M, f, x]$ with $f \in C_{\text{imm}}^\infty(M, \mathbb{E})$. The restriction of $\widehat{\mathcal{F}}_*(n)$ to $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is denoted by $\widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$. Observe that the canonical projection $M \rightarrow \text{Iso}(M, f) \setminus M$ is a covering map if $f \in C_{\text{imm}}^\infty(M, \mathbb{E})$.

On the other hand, let $\widehat{\mathcal{M}}_{*,c}^\infty(n)$ (respectively, $\widehat{\mathcal{M}}_{*,o}^\infty(n)$) be the $\widehat{\mathcal{F}}_*(n)$ -saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M, f, x]$ such that M is compact (respectively, open). Observe that, if $[N, h, y]$ is close enough to any $[M, f, x] \in \widehat{\mathcal{M}}_{*,c}^\infty(n)$, then N is diffeomorphic to M . Thus $\widehat{\mathcal{M}}_{*,c}^\infty(n)$ is open in $\widehat{\mathcal{M}}_*(n)$, and therefore $\widehat{\mathcal{M}}_{*,o}^\infty(n)$ is closed. Hence these are Polish subspaces of $\widehat{\mathcal{M}}_*(n)$, as well as their intersections with any Polish subspace. Let $\widehat{\mathcal{M}}_{*,\text{imm},c/o}^\infty(n) = \widehat{\mathcal{M}}_{*,c/o}^\infty(n) \cap \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$. The restrictions of $\widehat{\mathcal{F}}_*(n)$ to $\widehat{\mathcal{M}}_{*,c/o}^\infty(n)$ and $\widehat{\mathcal{M}}_{*,\text{imm},c/o}^\infty(n)$ are denoted by $\widehat{\mathcal{F}}_{*,c/o}^\infty(n)$ and $\widehat{\mathcal{F}}_{*,\text{imm},c/o}^\infty(n)$, respectively.

Theorem 1.4. *The following properties hold:*

- (i) $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is Polish and dense in $\widehat{\mathcal{M}}_*^\infty(n)$.
- (ii) $\widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$ is a foliated structure of dimension n .
- (iii) $\widehat{\mathcal{F}}_{*,\text{imm},o}^\infty(n)$ is transitive.
- (iv) There is a unique C^∞ foliated structure $\widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$ on $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$, whose underlying topological foliated structure is $\widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$, such that $\text{ev} : \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n) \rightarrow \mathbb{E}$ is a C^∞ immersion.
- (v) There is a unique Riemannian metric on $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n) \equiv (\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n), \widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n))$ such that $\iota_{M,f} : M \rightarrow \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is a local isometry for all complete connected Riemannian n -manifold M and $f \in C_{\text{imm}}^\infty(M, \mathbb{E})$.
- (vi) For all M and f as above, the map $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ is the holonomy covering of the leaf $\text{im } \hat{\iota}_{M,f}$.

It is possible to give a version of Theorem 1.4 closer to [1, Theorem 1.3], using the subspace $\widehat{\mathcal{M}}_{*,\text{inp}}^\infty(n)$ consisting of the classes $[M, f, x]$ such that $M \rightarrow \text{Iso}(M, f) \setminus M$ is a covering map. Such a result could be proved with the obvious adaptation of the proof of [1, Theorem 1.3], using the exponential map to define foliated charts. Instead, we have opted for studying $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ because, in this case, the immersions f directly provide foliated charts.

The following result states that $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is universal among the class of Polish Riemannian foliated spaces that satisfy a condition called covering-continuity (Definition 6.1).

Theorem 1.5. *A Polish Riemannian foliated space X of dimension n with complete leaves is isometric to a saturated Riemannian foliated subspace of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ if and only if X is covering-continuous.*

In Theorem 1.5, when X consists of a single leaf M , the isometric injection of M into $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is $\hat{\iota}_{M,f}$ for any C^∞ embedding $f : M \rightarrow \mathbb{E}$. If moreover M is of bounded geometry, then f can be chosen so that $\widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$ is a compact Riemannian foliated subspace of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ (Proposition 7.1). Then Theorem 1.1 follows by considering the isometric injection $\hat{\iota}_{M,f} : M \rightarrow \widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$.

There are examples of Lie groups with left invariant metrics that are not coarsely quasi-isometric to any finitely generated group [7], [11]. Applying the above argument to those Riemannian manifolds, we get

compact Riemannian foliated spaces whose leaf holonomy covers are not coarsely quasi-isometric to any finitely generated group.

Theorem 1.1 contrasts with the examples of connected Riemannian manifolds of bounded geometry whose quasi-isometry type cannot be realized as leaves of foliations of codimension one on closed manifolds [2], [37], [30], [31]. If the metric is not considered, any surface can be realized as a leaf of a codimension one foliation on a closed manifold [6], but this fails in higher dimension [12], [19], [2], [35], [32]. The study of this realizability problem was initiated in [34].

This work can be considered as a continuation of [1], and therefore many references to [1] are included.

2. PRELIMINARIES

Let M be a Riemannian manifold (possibly with boundary or corners). The following standard notation will be used. The metric tensor is denoted by g , the distance function on each of the connected components of M by d , the tangent bundle by $\pi : TM \rightarrow M$, the Levi-Civita connection by ∇ , and the open and closed balls of center $x \in M$ and radius $r > 0$ by $B(x, r)$ and $\overline{B}(x, r)$, respectively. If needed, “ M ” will be added to all of the above notation as a subindex or superindex; when a family of Riemannian manifolds M_i is considered, we may add the subindex or superindex “ i ” instead of “ M_i ”. A covering space of M is assumed to be equipped with the lift of g .

For $m \in \mathbb{Z}^+$, let $T^{(m)}M = T \cdots TM$ (m times); we also set $T^{(0)}M = M$. If $l < m$, $T^{(l)}M$ is identified with a regular submanifold of $T^{(m)}M$ via zero sections, and therefore, for each $x \in M$, the notation x may be also used for the zero elements of $T_x M$, $T_x TM$, etc. Let $\pi : T^{(m)}M \rightarrow T^{(l)}M$ be the vector bundle projection given by composing the tangent bundle projections; in particular, we have $\pi : T^{(m)}M \rightarrow M$. Given any C^m map between Riemannian manifolds, $\phi : M \rightarrow N$, the induced map $T^{(m)}M \rightarrow T^{(m)}N$ will be denoted by $\phi_*^{(m)}$ (or simply ϕ_* if $m = 1$).

Hilbert manifolds are also considered in some parts of the paper, using analogous notation.

The Levi-Civita connection determines a decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$, as direct sum of the horizontal and vertical subbundles. The *Sasaki metric* on TM is the unique Riemannian metric $g^{(1)}$ so that $\mathcal{H} \perp \mathcal{V}$ and the canonical identities $\mathcal{H}_\xi \equiv T_\xi M \equiv \mathcal{V}_\xi$ are isometries for every $\xi \in TM$ [27]. Continuing by induction, for $m \geq 2$, the *Sasaki metric* on $T^{(m)}M$ is $g^{(m)} = (g^{(m-1)})^{(1)}$. The notation $d^{(m)}$ is used for the corresponding distance function on the connected components, and the corresponding open and closed balls of center $\xi \in T^{(m)}M$ and radius $r > 0$ are denoted by $B^{(m)}(\xi, r)$ and $\overline{B}^{(m)}(\xi, r)$, respectively. We may add the subindex “ M ” to this notation if necessary, or the subindex “ i ” instead of “ M_i ” for a family of Riemannian manifolds M_i . From now on, $T^{(m)}M$ is assumed to be equipped with $g^{(m)}$. For $l < m$, $T^{(l)}M$ becomes a totally geodesic Riemannian submanifold of $T^{(m)}M$ orthogonal to the fibers of $\pi : T^{(m)}M \rightarrow T^{(l)}M$, which are also totally geodesic [1, Remark 1-(i)–(iii)] (see also [27, Corollary of Theorem 13, and Theorems 14 and 18]).

Let $(U; x^1, \dots, x^n)$ be a chart of M . As usual, the corresponding metric coefficients are denoted by g_{ij} , and write $(g^{ij}) = (g_{ij})^{-1}$. Identify the functions x^i with their lifts to TU . We get a chart $(U^{(1)}; x_{(1)}^1, \dots, x_{(1)}^{2n})$ of TM with $U^{(1)} = TU$, $x_{(1)}^i = x^i$ and $x_{(1)}^{n+i} = v^i$ for $1 \leq i \leq n$, where the functions v^i give the coordinates of tangent vectors with respect to the local frame $(\partial_1, \dots, \partial_n)$ of TU induced by $(U; x^1, \dots, x^n)$. By induction, for $m \geq 2$, let $(U^{(m)}; x_{(m)}^1, \dots, x_{(m)}^{2^m n})$ be the chart of $T^{(m)}M$ induced by the chart $(U^{(m-1)}; x_{(m-1)}^1, \dots, x_{(m-1)}^{2^{m-1} n})$ of $T^{(m-1)}M$.

Let $\Omega \subset M$ be a compact domain and $m \in \mathbb{N}$. Fix a finite collection of charts of M that covers Ω , $\mathcal{U} = \{(U_a; x_a^1, \dots, x_a^n)\}$, and a family of compact subsets of M with the same index set as \mathcal{U} , $\mathcal{K} = \{K_a\}$, such that $\Omega \subset \bigcup_a K_a$, and $K_a \subset U_a$ for all a . The corresponding C^m norm of a C^m tensor T on Ω is defined by⁴

$$\|T\|_{C^m, \Omega, \mathcal{U}, \mathcal{K}} = \max_a \max_{x \in K_a \cap \Omega} \sum_{|I| \leq m} \sum_{J, K} \left| \frac{\partial^{|I|} T_{a, J}^K}{\partial x_a^I}(x) \right|,$$

where $T_{a, J}^K$ are the coefficients of T on $U_a \cap \Omega$ with respect to the frame induced by $(U_a; x_a^1, \dots, x_a^n)$. With this norm, the C^m tensors on Ω of a fixed type form a Banach space, whose underlying topology is called

⁴The standard multi-index notation is used here.

the C^m topology. By taking the projective limit as $m \rightarrow \infty$, we get the Fréchet space of C^∞ tensors of that type, whose underlying topology is called the C^∞ topology (see e.g. [18]). We will always consider the C^k topology for C^k tensors on Ω of a given type ($k \in \mathbb{N} \cup \{\infty\}$); in particular, $C^k(\Omega)$ is always assumed to be equipped with the C^k topology. Observe that \mathcal{U} and \mathcal{K} are also qualified to define the norm $\| \cdot \|_{C^m, \Omega', \mathcal{U}, \mathcal{K}}$ for any compact subdomain $\Omega' \subset \Omega$. It is well known that $\| \cdot \|_{C^m, \Omega, \mathcal{U}, \mathcal{K}}$ is equivalent to the norm $\| \cdot \|_{C^m, \Omega, g}$ defined by

$$\|T\|_{C^m, \Omega, g} = \max_{0 \leq l \leq m} \max_{x \in \Omega} |\nabla^l T(x)| ;$$

i.e., there is some $C \geq 1$, depending on $M, \Omega, \mathcal{U}, \mathcal{K}, g$ and m , such that

$$\frac{1}{C} \| \cdot \|_{C^m, \Omega, \mathcal{U}, \mathcal{K}} \leq \| \cdot \|_{C^m, \Omega, g} \leq C \| \cdot \|_{C^m, \Omega, \mathcal{U}, \mathcal{K}} . \quad (1)$$

In particular, for $m = 0$ and $f \in C^\infty(M)$,

$$\|f\|_\Omega := \|f\|_{C^0, \Omega, \mathcal{U}, \mathcal{K}} = \|f\|_{C^0, \Omega, g} = \max_{x \in \Omega} |f(x)| , \quad (2)$$

which is independent of the choices \mathcal{U}, \mathcal{K} and g .

The norms $\| \cdot \|_{C^m, \Omega, \mathcal{U}, \mathcal{K}}$ and $\| \cdot \|_{C^m, \Omega, g}$ have straightforward extensions to tensors with values in a separable Hilbert space \mathbb{E} , and satisfy the obvious versions of (1) and (2), and $C^k(M, \mathbb{E})$ is assumed to be equipped with the C^k topology ($k \in \mathbb{N} \cup \{\infty\}$).

For $f \in C^\infty(M, \mathbb{E})$, recall that $\nabla f = df$ (its de Rham differential). For each m , the map

$$f_*^{(m)} \equiv \left(f_*^{(m), 1}, \dots, f_*^{(m), 2^m} \right) : T^{(m)}M \rightarrow T^{(m)}\mathbb{E} \equiv \mathbb{E}^{2^m}$$

is also C^∞ and with values in a separable Hilbert space. In the following lemma, we consider the local representations of f and every $f_*^{(m), \lambda}$ with respect to coordinate systems (U, x^1, \dots, x^n) and $(U^{(m)}, x_{(m)}^1, \dots, x_{(m)}^{2^m n})$ of M and $T^{(m)}M$. Moreover each function on M or U is identified with its lift to $T^{(m)}M$ or $U^{(m)}$.

Lemma 2.1. *The following properties hold:*

- (i) *The local representation of every $f_*^{(m), \lambda}$ is a universal polynomial expression of $x_{(m)}^{n+1}, \dots, x_{(m)}^{2^m n}$ and the partial derivatives up to order m of the local representation of f .*
- (ii) *For each $\rho > 0$, the partial derivatives up to order m of the local representation of f are given by universal linear expressions of the functions $(\sigma_{\rho, \mu}^{(m)})^* f_*^{(m), \lambda}$ for $n+1 \leq \mu \leq 2^m n$, where $\sigma_{\rho, \mu}^{(m)} : U \rightarrow U^{(m)}$ is the section of $\pi : U^{(m)} \rightarrow U$ determined by⁵ $(\sigma_{\rho, \mu}^{(m)})^* x_{(m)}^\nu = \rho \delta_{\mu\nu}$ for $n+1 \leq \nu \leq 2^m n$.*

Proof. By using induction on m , the result clearly boils down to the case $m = 1$. But, in this case, the statement follows because $f_* \equiv (f, df) : TM \rightarrow T\mathbb{E} \equiv \mathbb{E}^2$. \square

By using the supremum on Ω instead of the maximum, the definition of $\| \cdot \|_{C^m, \Omega, g}$ can be extended to any non-compact n -submanifold $\Omega \subset M$ (including $\Omega = M$), with possible infinite values. The tensors on Ω with finite norm $\| \cdot \|_{C^m, \Omega, g}$ are said to be *uniformly C^m* , or C_b^m . For a given type, they form a Banach space, and the corresponding projective limit as $m \rightarrow \infty$ is a Fréchet space, whose elements are said to be *uniformly C^∞* , or C_b^∞ (see e.g. [26, Definition 2.7] or [28, Definition 3.15]). In particular, this gives rise to the Fréchet spaces $C_b^\infty(\Omega)$ and $C_b^\infty(\Omega, \mathbb{E})$ when \mathbb{R} -valued and \mathbb{E} -valued C_b^∞ functions are considered.

Let N be another Riemannian manifold. Recall that a C^1 map $\phi : M \rightarrow N$ is called a (λ) *quasi-isometry*, or (λ) *quasi-isometric*, if there is some $\lambda \geq 1$ such that $\frac{1}{\lambda} |\xi| \leq |\phi_*(\xi)| \leq \lambda |\xi|$ for every $\xi \in TM$; in particular, ϕ is an immersion. To define higher order quasi-isometries, let $T^{\leq r} M = \{ \xi \in TM \mid |\xi| \leq r \}$ for each $r > 0$. If M has no boundary, then $T^{\leq r} M$ is a manifold with boundary; otherwise, it is a manifold with corners. Also, define $T^{(m), \leq r} M$ by induction on $m \in \mathbb{Z}^+$, setting $T^{(1), \leq r} M = T^{\leq r} M$ and $T^{(m), \leq r} M = T^{\leq r} T^{(m-1), \leq r} M$. It is said that $\phi : M \rightarrow N$ is a (λ) *quasi-isometry of order $m \in \mathbb{N}$* , or a (λ) *quasi-isometric map of order m* , if it is C^{m+1} and $\phi_*^{(m)} : T^{(m), \leq 1} M \rightarrow T^{(m)} N$ is a (λ) quasi-isometry. If ϕ is a quasi-isometry of order m for all $m \in \mathbb{N}$, then it is called a *quasi-isometry of order ∞* . If there is a quasi-isometric diffeomorphism $M \rightarrow N$ of order $m \in \mathbb{N} \cup \{\infty\}$, then M and N are said to be *quasi-isometric with order m* . The property of being a quasi-isometry of order m is preserved by the operations of composition of maps and inversion of

⁵Kronecker's delta is used here.

diffeomorphisms [1, Proposition 3.9], and therefore it induces an equivalence relation between Riemannian manifolds.

For $m \in \mathbb{N}$, a partial map $\phi : M \rightharpoonup N$ is called a C^m local diffeomorphism⁶ if $\text{dom } \phi$ and $\text{im } \phi$ are open in M and N , respectively, and $\phi : \text{dom } \phi \rightarrow \text{im } \phi$ is a C^m diffeomorphism. If moreover $\phi(x) = y$ for distinguished points, $x \in \text{dom } \phi$ and $y \in \text{im } \phi$, then it is said that $\phi : (M, x) \rightharpoonup (N, y)$ is a *pointed C^m local diffeomorphism*. For $m \in \mathbb{N}$, $R > 0$ and $\lambda \geq 1$, a C^{m+1} pointed local diffeomorphism $\phi : (M, x) \rightharpoonup (N, y)$ is called an (m, R, λ) -pointed local quasi-isometry, or a *local quasi-isometry of type (m, R, λ)* , if the restriction $\phi_*^{(m)} : \Omega^{(m)} \rightarrow T^{(m)}N$ is a λ -quasi-isometry for some compact domain $\Omega^{(m)} \subset \text{dom } \phi_*^{(m)}$ with $B_M^{(m)}(x, R) \subset \Omega^{(m)}$ [1, Definition 4.2].

3. (PARTIAL) QUASI-EQUIVALENCES

Let M and N be Riemannian n -manifolds, let $f \in C^\infty(M, \mathbb{E})$ and $h \in C^\infty(N, \mathbb{E})$, and let $x \in M$ and $y \in N$. Recall from Section 1 the concepts of an equivalence $(M, f) \rightarrow (N, h)$, and a pointed equivalence $(M, f, x) \rightarrow (N, h, y)$. Observe that $\|f_*^{(m)}\|_{\Omega^{(m)}}$ makes sense for any n -submanifold $\Omega^{(m)} \subset T^{(m)}M$ because we consider $f_*^{(m)} : T^{(m)}M \rightarrow T^{(m)}\mathbb{E} \equiv \mathbb{E}^{2^m}$, with values in a separable Hilbert space. Note also that $(\phi^*h)_*^{(m)} = h_*^{(m)} \circ \phi_*^{(m)}$ for any C^m map $\phi : M \rightarrow N$.

Definition 3.1. Let $\lambda \geq 1$ and $\varepsilon \geq 0$, and let $\phi : M \rightarrow N$ be a C^1 map. It is said that $\phi : (M, f) \rightarrow (N, h)$ is a $((\lambda, \varepsilon)$ -) *quasi-equivalence of order $m \in \mathbb{N}$* if it is C^{m+1} , $\phi_*^{(m)} : T^{(m), \leq 1}M \rightarrow T^{(m)}N$ is a (λ) -quasi-isometry, and $\|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{T^{(m)}M} \leq \varepsilon$. If moreover distinguished points x and y are preserved, then $\phi : (M, f, x) \rightarrow (N, h, y)$ is called a *pointed quasi-equivalence of order m* . If there is a quasi-equivalence $(M, f) \rightarrow (N, h)$ (respectively, $(M, f, x) \rightarrow (N, h, y)$), then (M, f) and (N, h) (respectively, (M, f, x) and (N, h, y)) are called *quasi-equivalent*.

Remark 1. (i) Any (λ, ε) -quasi-equivalence of order $m \geq 1$ is a (λ, ε) -quasi-equivalence of order $m - 1$.
(ii) For integers $0 \leq m' \leq m$, if ϕ is a (λ, ε) -quasi-equivalence of order m , then $\phi_*^{(m')}$ is a (λ, ε) -quasi-equivalence of order $m - m'$.

For a submanifold $\Omega \subset M$ and $f \in C^\infty(M, \mathbb{E})$, the notation (Ω, f) is used for $(\Omega, f|_\Omega)$.

Proposition 3.2. *The following properties hold for any $m \in \mathbb{N}$, $\lambda, \mu \geq 1$ and $\varepsilon, \delta \geq 0$:*

- (i) *There is some $\nu \geq 1$, depending on m , λ and μ , such that, if $\phi : (M, f) \rightarrow (N, h)$ is a (λ, ε) -quasi-equivalence and $\psi : (N, h) \rightarrow (L, u)$ a (μ, δ) -quasi-equivalence, both of them of order m , then $\psi \circ \phi : (M, f) \rightarrow (L, u)$ is a $(\nu, \varepsilon + \delta)$ -quasi-equivalence of order m .*
- (ii) *There are some $\nu' \geq 1$, depending on m and λ , such that, if $\phi : (M, f) \rightarrow (N, h)$ is a (λ, ε) -quasi-equivalence of order m and a diffeomorphism, then $\phi^{-1} : (N, h) \rightarrow (M, f)$ is a (ν', ε) -quasi-equivalence of order m .*

Proof. By [1, Proposition 3.9], we only have to check the conditions on the \mathbb{E} -valued functions. Thus (i) follows because, for each $\xi \in T^{(m)}M$, we have

$$\begin{aligned} & \left\| f_*^{(m)}(\xi) - ((\psi \circ \phi)_* u)_*^{(m)}(\xi) \right\| \\ & \leq \left\| f_*^{(m)}(\xi) - (\phi^*h)_*^{(m)}(\xi) \right\| + \left\| h_*^{(m)}\left(\phi_*^{(m)}(\xi)\right) - (\psi^*u)_*^{(m)}\left(\phi_*^{(m)}(\xi)\right) \right\| \leq \varepsilon + \delta. \end{aligned}$$

Similarly, (ii) follows because, for each $\zeta \in T^{(m)}N$,

$$\left\| h_*^{(m)}(\zeta) - ((\phi^{-1})^* f)_*^{(m)}(\zeta) \right\| = \left\| (\phi^*h)_*^{(m)}\left((\phi^{-1})_*^{(m)}(\zeta)\right) - f_*^{(m)}\left((\phi^{-1})_*^{(m)}(\zeta)\right) \right\| \leq \varepsilon. \quad \square$$

Corollary 3.3. *“Being quasi-equivalent with order m ” is an equivalence relation on the sets of pairs (M, f) and triples (M, f, x) .*

Now, suppose that M and N are connected, complete and without boundary.

⁶The term “ C^m local diffeomorphism” ($m \geq 1$) is also used in the standard sense, referring to any C^m map $M \rightarrow N$ whose tangent map is an isomorphism at every point of M . The context will always clarify this ambiguity.

Definition 3.4. Fix $m \in \mathbb{N}$, $R > 0$, $\lambda \geq 1$ and $\varepsilon \geq 0$. Let $\phi : (M, x) \rightarrow (N, y)$ be a C^{m+1} pointed local diffeomorphism, and let $f \in C^\infty(M, \mathbb{E})$ and $h \in C^\infty(N, \mathbb{E})$. It is said that $\phi : (M, f, x) \rightarrow (N, h, y)$ is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence, or a local quasi-equivalence of type $(m, R, \lambda, \varepsilon)$, if there is some compact domain $\Omega^{(m)} \subset \text{dom } \phi_*^{(m)}$ such that $B_M^{(m)}(x, R) \subset \Omega^{(m)}$ and $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \rightarrow (T^{(m)}N, h_*^{(m)})$ is a (λ, ε) -quasi-equivalence.

Remark 2. (i) Any pointed local quasi-equivalence $(M, f, x) \rightarrow (N, h, y)$ of type $(m, R, \lambda, \varepsilon)$ is also of type $(m', R', \lambda', \varepsilon')$ for $0 \leq m' \leq m$, $0 < R' < R$, $\lambda' > \lambda$ and $\varepsilon' > \varepsilon$.
(ii) Consider integers $0 \leq m' \leq m$, any pointed C^{m+1} local diffeomorphism $\phi : (M, x) \rightarrow (N, y)$, and any $f \in C^\infty(M, \mathbb{E})$ and $h \in C^\infty(N, \mathbb{E})$. Then $\phi : (M, f, x) \rightarrow (N, h, y)$ is a pointed local quasi-equivalence of type $(m, R, \lambda, \varepsilon)$ if and only if $\phi_*^{(m')} : (T^{(m')}M, f_*^{(m')}, x) \rightarrow (T^{(m')}N, h_*^{(m')}, y)$ is a pointed local quasi-equivalence of type $(m - m', R, \lambda, \varepsilon)$.
(iii) If there is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence $(M, f, x) \rightarrow (N, h, y)$, then, for all $R' < R$, $\lambda' > \lambda$ and $\varepsilon' > \varepsilon$, there is a C^∞ $(m, R', \lambda', \varepsilon')$ -pointed local quasi-equivalence $(M, f, x) \rightarrow (N, h, y)$ by [18, Theorem 2.7].

Lemma 3.5. *The following properties hold:*

- (i) If $\phi : (M, f, x) \rightarrow (N, h, y)$ and $\psi : (N, h, y) \rightarrow (L, u, z)$ are pointed local quasi-equivalences of types $(m, R, \lambda, \varepsilon)$ and $(m, \lambda R, \lambda', \varepsilon')$, respectively, then $\psi \circ \phi : (M, f, x) \rightarrow (L, u, z)$ is an $(m, R, \lambda\lambda', \varepsilon + \varepsilon')$ -pointed local quasi-equivalence.
- (ii) If $\phi : (M, f, x) \rightarrow (N, h, y)$ is an $(m, \lambda R, \lambda, \varepsilon)$ -pointed local quasi-isometry, then $\phi^{-1} : (N, h, y) \rightarrow (M, f, x)$ is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-isometry.

Proof. To prove (i), take compact domains, $\Omega^{(m)} \subset T^{(m)}M$ and $\Omega'^{(m)} \subset T^{(m)}N$, such that $B_M^{(m)}(x, R) \subset \Omega^{(m)}$, $B_N^{(m)}(x, \lambda R) \subset \Omega'^{(m)}$, $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \rightarrow (T^{(m)}N, h_*^{(m)})$ is a (λ, ε) -quasi-equivalence, and $\psi_*^{(m)} : (\Omega'^{(m)}, h_*^{(m)}) \rightarrow (T^{(m)}L, u_*^{(m)})$ is a (λ', ε') -quasi-equivalence. According to the proof of [1, Lemma 4.3-(i)], there is a compact domain $\Omega_0^{(m)} \subset T^{(m)}M$ such that $B_M^{(m)}(x, R) \subset \Omega_0^{(m)}$ and $\phi_*^{(m)}(\Omega_0^{(m)}) \subset \Omega'^{(m)}$. Then $(\psi \circ \phi)_*^{(m)} : \Omega_0^{(m)} \rightarrow T^{(m)}L$ is a $\lambda\lambda'$ -quasi-isometry by [1, Remark 2-(v)]. Moreover, for each $\xi \in \Omega_0^{(m)}$,

$$\begin{aligned} & \left\| f_*^{(m)}(\xi) - ((\psi \circ \phi)_* u)_*^{(m)}(\xi) \right\| \\ & \leq \left\| f_*^{(m)}(\xi) - (\phi^* h)_*^{(m)}(\xi) \right\| + \left\| h_*^{(m)} \left(\phi_*^{(m)}(\xi) \right) - (\psi^* u)_*^{(m)} \left(\phi_*^{(m)}(\xi) \right) \right\| \leq \varepsilon + \varepsilon'. \end{aligned}$$

So $\psi \circ \phi : (M, f, x) \rightarrow (L, u, z)$ is an $(m, R, \lambda\lambda', \varepsilon + \varepsilon')$ -pointed local quasi-equivalence.

To prove (ii), let $\Omega^{(m)} \subset T^{(m)}M$ be a compact domain such that $B_M^{(m)}(x, R) \subset \Omega^{(m)}$, and $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \rightarrow (T^{(m)}N, h_*^{(m)})$ is a (λ, ε) -quasi-equivalence. According to the proof of [1, Lemma 4.3-(ii)], the compact domain $\Omega'^{(m)} := \phi_*^{(m)}(\Omega^{(m)}) \subset T^{(m)}N$ contains $B_N^{(m)}(y, R)$. Then $(\phi^{-1})_*^{(m)} = (\phi_*^{(m)})^{-1} : \Omega'^{(m)} \rightarrow T^{(m)}M$ is a λ -quasi-isometry by [1, Remark 2-(vi)]. Moreover, for each $\xi \in \Omega'^{(m)}$,

$$\left\| h_*^{(m)}(\xi) - ((\phi^{-1})^* f)_*^{(m)}(\xi) \right\| \leq \left\| (\phi^* h)_*^{(m)} \left((\phi^{-1})_*^{(m)}(\xi) \right) - f_*^{(m)} \left((\phi^{-1})_*^{(m)}(\xi) \right) \right\| \leq \varepsilon.$$

So $\phi^{-1} : (N, h, y) \rightarrow (M, f, x)$ is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence. \square

4. THE C^∞ TOPOLOGY ON $\widehat{\mathcal{M}}_*(n)$

Definition 4.1. For $m \in \mathbb{N}$ and $R, r > 0$, let $\widehat{U}_{R,r}^m$ be the set of pairs $([M, f, x], [N, h, y]) \in \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ such that there is some $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence $(M, f, x) \rightarrow (N, h, y)$ for some $\lambda \in [1, e^r]$ and $\varepsilon \in (0, r)$.

Proposition 4.2. *The following properties⁷ hold for all $m, m' \in \mathbb{N}$ and $R, S, r, s > 0$:*

⁷The following standard notation is used for a set X and relations $U, V \subset X \times X$:

$$U^{-1} = \{ (y, x) \in X \times X \mid (x, y) \in U \},$$

$$V \circ U = \{ (x, z) \in X \times X \mid \exists y \in X \text{ so that } (x, y) \in U \text{ and } (y, z) \in V \}.$$

Moreover the diagonal of $X \times X$ is denoted by Δ .

- (i) $(\widehat{U}_{e^r R, r}^m)^{-1} \subset \widehat{U}_{R, r}^m$.
- (ii) $\widehat{U}_{R_0, r_0}^{m_0} \subset \widehat{U}_{R, r}^m \cap \widehat{U}_{S, s}^{m'}$, where $m_0 = \max\{m, m'\}$, $R_0 = \max\{R, S\}$ and $r_0 = \min\{r, s\}$.
- (iii) $\Delta \subset \widehat{U}_{R, r}^m$.
- (iv) $\widehat{U}_{R, r}^m \circ \widehat{U}_{e^r R, s}^m \subset \widehat{U}_{R, r+s}^m$.

Proof. Properties (ii) and (iii) are elementary, and (i) and (iv) are consequences of Lemma 3.5. \square

Proposition 4.3. $\bigcap_{R, r > 0} \widehat{U}_{R, r}^m = \Delta$ for all $m \in \mathbb{N}$.

Proof. We only have to prove “ \subset ” by Proposition 4.2-(iii). For $([M, f, x], [N, h, y]) \in \bigcap_{R, r > 0} \widehat{U}_{R, r}^m$, there is a sequence of pointed local quasi-equivalences $\phi_i : (M, f, x) \rightarrow (N, h, y)$, with corresponding types $(m, R_i, \lambda_i, \varepsilon_i)$, such that $R_i \uparrow \infty$, $\lambda_i \downarrow 1$ and $\varepsilon_i \downarrow 0$ as $i \rightarrow \infty$. According to the proof of [1, Proposition 5.3], for each i , there is some subsequence $\phi_{k(i, l)}$ whose restriction to $B_M(x, R_i)$ converges to some pointed isometric immersion $\psi_i : (B_M(x, R_i), x) \rightarrow (N, y)$ in the weak C^m topology, $\psi_{i+1}|_{B_M(x, R_i)} = \psi_i$ for all i , and the combination of the maps ψ_i is a pointed isometry $\psi : (M, x) \rightarrow (N, y)$. For every $x' \in M$ and $\varepsilon > 0$, there are some i and $\delta > 0$ so that $x' \in B_M(x, R_i)$, $\varepsilon_i \leq \varepsilon/2$, and $\|h(y') - h(y'')\| < \varepsilon/2$ if $d_N(y', y'') < \delta$ for all $y', y'' \in \overline{B}_M(x, R_i)$. Moreover there is some l such that $d_N(\phi_{k(i, l)}(x'), \psi_i(x')) < \delta$. Hence

$$\|f(x') - h \circ \psi(x')\| \leq \|f(x') - h \circ \phi_{k(i, l)}(x')\| + \|h \circ \phi_{k(i, l)}(x') - h \circ \psi(x')\| < \varepsilon_i + \varepsilon/2 \leq \varepsilon.$$

Since x' and ε are arbitrary, it follows that $\psi : (M, f, x) \rightarrow (N, h, y)$ is an equivalence, and therefore $[M, f, x] = [N, h, y]$. \square

By Propositions 4.2 and 4.3, the sets $\widehat{U}_{R, r}^m$ form a base of entourages of a separating uniformity on $\widehat{\mathcal{M}}_*(n)$, which is called the C^∞ uniformity.

Definition 4.4. For $R, r > 0$ and $m \in \mathbb{N}$, let $\widehat{D}_{R, r}^m$ be the set of pairs $([M, f, x], [N, h, y]) \in \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ such that there is some C^{m+1} pointed local diffeomorphism $\phi : (M, x) \rightarrow (N, y)$ so that $\|g_M - \phi^* g_N\|_{C^m, \Omega, g_M} < r$ and $\|f - \phi^* h\|_{C^m, \Omega, g_M} < r$ for some compact domain $\Omega \subset \text{dom } \phi$ with $B_M(x, R) \subset \Omega$.

Remark 3. By (1), and its version for \mathbb{E} -valued functions, a sequence $[M_i, f_i, x_i] \in \widehat{\mathcal{M}}_*(n)$ is C^∞ convergent to $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$ if and only if it is eventually in $\widehat{D}_{R, r}^m(M, f, x)$ for arbitrary $m \in \mathbb{N}$ and $R, r > 0$.

Proposition 4.5. *The following properties hold:*

- (i) For all $R, r > 0$, if $0 < r' \leq \min\{1 - e^{-2r}, e^{2r} - 1, r\}$, then $\widehat{D}_{R, r'}^0 \subset \widehat{U}_{R, r}^0$.
- (ii) For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$, there is some $r' > 0$ such that $\widehat{D}_{R, r'}^m(M, f, x) \subset \widehat{U}_{R, r}^m(M, f, x)$.

Proof. Let us show (i). If $([M, f, x], [N, h, y]) \in \widehat{D}_{R, r'}^0$, then there is a C^1 pointed local diffeomorphism $\phi : (M, x) \rightarrow (N, y)$ such that $r'_0 := \|g_M - \phi^* g_N\|_{C^0, \Omega, g_M} < r'$ and $\varepsilon_0 := \|f - \phi^* h\|_{C^0, \Omega, g_M} < r'$ for some compact domain $\Omega \subset \text{dom } \phi$ with $B_M(x, R) \subset \Omega$. Take some $\lambda \in [1, e^r]$ such that $r'_0 \leq \min\{1 - \lambda^{-2}, \lambda^2 - 1\}$. According to the proof of [1, Proposition 6.4-(i)], $\phi : \Omega \rightarrow N$ is a λ -quasi-isometry. Since moreover $\|f - \phi^* h\|_\Omega \leq \varepsilon_0$, it follows that ϕ is a $(0, R, \lambda, r'_0, \varepsilon_0)$ -pointed local quasi-equivalence, obtaining that $([M, f, x], [N, h, y]) \in \widehat{U}_{R, r}^0$.

Let us prove (ii). Take $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$. Let \mathcal{U} be a finite collection of charts of M with domains U_a , and let $\mathcal{K} = \{K_a\}$ be a family of compact subsets of M , with the same index set as \mathcal{U} , such that $K_a \subset U_a$ for all a , and $\overline{B}_M(x, R) \subset \text{Int}(K)$ for $K = \bigcup_a K_a$. Let $r' > 0$, to be fixed later. For any $[N, h, y] \in \widehat{D}_{R, r'}^m(M, x)$, there is a C^{m+1} pointed local diffeomorphism $\phi : (M, x) \rightarrow (N, y)$ so that $\|g_M - \phi^* g_N\|_{C^m, \Omega, g_M} < r'$ and $\|f - \phi^* h\|_{C^m, \Omega, g_M} < r'$ for some compact domain $\Omega \subset \text{dom } \phi \cap \text{Int}(K)$ with $B_M(x, R) \subset \Omega$. By continuity, there is another compact domain $\Omega' \subset \text{dom } \phi \cap \text{Int}(K)$ such that $\Omega \subset \text{Int}(\Omega')$, $\|g_M - \phi^* g_N\|_{C^m, \Omega', g_M} < r'$ and $\|f - \phi^* h\|_{C^m, \Omega', g_M} < r'$. According to the proof of [1, Proposition 6.4-(i)], if r' is small enough (depending on m, R, r and $[M, x]$), then there is some compact domain $\Omega^{(m)} \subset T^{(m)} M$

⁸Given a set X , for $U \subset X \times X$ and $x \in X$, let $U(x) = \{y \in Y \mid (x, y) \in U\}$. In the case of $U \subset \widehat{\mathcal{M}}_*(n) \times \widehat{\mathcal{M}}_*(n)$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$, we simply write $U(M, f, x)$.

such that $B_M^{(m)}(x, R) \subset \Omega^{(m)} \subset \pi^{-1}(\Omega')$, where $\pi : T^{(m)}M \rightarrow M$, and $\phi_*^{(m)} : \Omega^{(m)} \rightarrow T^{(m)}N$ is a λ -quasi-isometry for some $\lambda \in [1, e^r)$. Given $\varepsilon \in (0, r)$, choose some $C \geq 1$ satisfying (1) for \mathbb{E} -valued functions with \mathcal{U} , \mathcal{K} , Ω' and g , and, according to Lemma 2.1-(i), choose some $\varepsilon' > 0$ such that

$$\|f - \phi^*h\|_{C^m, \Omega', \mathcal{U}, \mathcal{K}} < \varepsilon' \implies \|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{\Omega^{(m)}} < \varepsilon.$$

Suppose that $r' \leq \varepsilon'/C$. Then

$$\|f - \phi^*h\|_{C^m, \Omega', g_M} < r' \implies \|f - \phi^*h\|_{C^m, \Omega', \mathcal{U}, \mathcal{K}} < Cr' \leq \varepsilon' \implies \|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{\Omega^{(m)}} < \varepsilon.$$

Hence ϕ is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence $(M, f, x) \mapsto (N, h, y)$, and therefore $[N, h, y] \in \widehat{U}_{R,r}^{(m)}(M, f, x)$. \square

Proposition 4.6. *The following properties hold:*

- (i) For all $R, r > 0$, if $e^{2r'} - e^{-2r'} \leq r$, then $\widehat{U}_{R,r'}^0 \subset \widehat{D}_{R,r}^0$.
- (ii) For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$, there is some $r' > 0$ such that $\widehat{U}_{R,r'}^m(M, f, x) \subset \widehat{D}_{R,r}^m(M, f, x)$.

Proof. Let us show (i). If $([M, f, x], [N, h, y]) \in \widehat{U}_{R,r'}^0$, then there is a $(0, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence $\phi : (M, f, x) \mapsto (N, h, y)$ for some $\lambda \in [1, e^{r'})$ and $\varepsilon \in (0, r')$. Thus there is some compact domain $\Omega \subset \text{dom } \phi$ such that $B_M(x, R) \subset \Omega$ and $\phi : (\Omega, f) \rightarrow (N, h)$ is a (λ, ε) -quasi-equivalence. According to the proof of [1, Proposition 6.5-(i)], $\|g_M - \phi^*g_N\|_{C^0, \Omega, g} < r$. So $([M, f, x], [N, h, y]) \in \widehat{D}_{R,r}^0$.

Let us prove (ii). Let $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M, f, x] \in \widehat{\mathcal{M}}_*(n)$. Take \mathcal{U} , \mathcal{K} and K like in the proof of Proposition 4.5-(ii). Let $r' > 0$, to be fixed later. For any $[N, h, y] \in \widehat{U}_{R,r'}^m(M, x)$, there is an $(m, R, \lambda, \varepsilon)$ -pointed local quasi-equivalence $\phi : (M, f, x) \mapsto (N, h, y)$ for some $\lambda \in [1, e^{r'})$ and $\varepsilon \in (0, r')$. Thus there is a compact domain $\Omega^{(m)} \subset \text{dom } \phi_*^{(m)} \cap \text{Int}(K^{(m)})$ so that $B_M^{(m)}(x, R) \subset \Omega^{(m)}$ and $\phi_*^{(m)} : (\Omega^{(m)}, f_*^{(m)}) \rightarrow (T^{(m)}N, h_*^{(m)})$ is a (λ, ε) -quasi-equivalence. According to the proof of [1, Proposition 6.5-(ii)], there are compact domains, $\Omega'^{(m)} \subset \text{dom } \phi_*^{(m)}$ and $\Omega \subset M$, such that $\Omega^{(m)} \subset \text{Int}(\Omega'^{(m)})$, $\Omega^{(m)} \cap M \subset \Omega \subset \text{Int}(\Omega'^{(m)})$, and $\|g_M - \phi^*g_N\|_{C^m, \Omega, g} < r$ if r' is small enough; in particular, $B_M(x, R) \subset \Omega$ because M is a totally geodesic Riemannian submanifold of $T^{(m)}M$. Take some $C \geq 1$ satisfying (1) for \mathbb{E} -valued functions with \mathcal{U} , \mathcal{K} , Ω and g_M . With the notation of Section 2, for $\rho > 0$ and $n+1 \leq \mu \leq 2^m n$, let $\sigma_{a, \rho, \mu}^{(m)} : U_a \rightarrow U_a^{(m)}$ be the section of each $\pi : U_a^{(m)} \rightarrow U_a$ of the type used in Lemma 2.1-(ii). Since $\Omega \subset \text{Int}(\Omega'^{(m)})$, there is some $\rho > 0$ so that $\sigma_{\rho, \mu}^{(m)}(K_a \cap \Omega) \subset \Omega'^{(m)}$ for all a and μ . Thus, by Lemma 2.1-(ii), there is some $\varepsilon' > 0$, depending on r and ρ , such that

$$\|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{\Omega'^{(m)}} < \varepsilon' \implies \|f^* - \phi^*h\|_{C^m, \Omega, \mathcal{U}, \mathcal{K}} < r/C.$$

Suppose that moreover $r' < \varepsilon'$, and therefore $\varepsilon < \varepsilon'$. Then

$$\|f_*^{(m)} - (\phi^*h)_*^{(m)}\|_{\Omega'^{(m)}} \leq \varepsilon < \varepsilon' \implies \|f - \phi^*h\|_{C^m, \Omega, \mathcal{U}, \mathcal{K}} < r/C \implies \|f - \phi^*h\|_{C^m, \Omega, g} < r,$$

showing that $[N, h, y] \in \widehat{D}_{R,r}^{(m)}(M, f, x)$. \square

Corollary 4.7. *The C^∞ convergence in $\widehat{\mathcal{M}}_*(n)$ describes the topology induced by the C^∞ uniformity.*

Proof. This is a direct consequence of Remark 3 and Propositions 4.5 and 4.6. \square

According to Corollary 4.7, the C^∞ uniformity induces what was called the C^∞ topology in Section 1. Recall that the corresponding space is denoted by $\widehat{\mathcal{M}}_*^\infty(n)$, and the notation $\widehat{\text{Cl}}_\infty$ is used for the closure operator in $\widehat{\mathcal{M}}_*^\infty(n)$.

Proposition 4.8. *$\widehat{\mathcal{M}}_*^\infty(n)$ is separable.*

Proof. According to the proof of [1, Proposition 7.1], there is a countable family \mathcal{C} of C^∞ compact manifolds containing exactly one representative of every diffeomorphism class, and, for every $M \in \mathcal{C}$, there is a countable

dense subset \mathcal{G}_M of the space of metrics on M with the C^∞ topology. Take also countable dense subsets, $\mathcal{D}_M \subset M$ and $\mathcal{F}_M \subset C^\infty(M, \mathbb{E})$. Then, like in the proof of [1, Proposition 7.1], the countable set

$$\{[(M, g), f, x] \mid M \in \mathcal{C}, g \in \mathcal{G}_M, x \in \mathcal{D}_M, f \in \mathcal{F}_M\} \quad (3)$$

is dense in $\widehat{\mathcal{M}}_*^\infty(n)$. \square

Proposition 4.9. *The C^∞ uniformity is complete and metrizable.*

Proof. According to [36, Corollary 38.4], the C^∞ uniformity on $\widehat{\mathcal{M}}_*(n)$ is metrizable because it is separating and the sets $\widehat{U}_{k,1/k}$ ($k \in \mathbb{Z}^+$) form a countable base of entourages. To check that this uniformity is complete, consider an arbitrary Cauchy sequence $[M_i, f_i, x_i]$ in $\widehat{\mathcal{M}}_*(n)$. We have to prove that $[M_i, f_i, x_i]$ is convergent in $\widehat{\mathcal{M}}_*^\infty(n)$. By taking a subsequence if necessary, we can suppose that $([M_i, f_i, x_i], [M_{i+1}, x_{i+1}, f_{i+1}]) \in U_{R_i, r_i}^{m_i}$ for sequences, $m_i \uparrow \infty$ in \mathbb{N} , and $R_i \uparrow \infty$ and $r_i \downarrow 0$ in \mathbb{R}^+ , such that $\sum_i r_i < \infty$, and $R_{i+1} \geq e^{r_i} R_i$ for all i . Let $\bar{r}_i = \sum_{j \geq i} r_j$. Consider other sequences $R'_i, R''_i \uparrow \infty$ in \mathbb{R}^+ such that $R'_i < R''_i \leq e^{-\bar{r}_i} R_i$ and $R'_{i+1} \geq e^{r_i} R''_i$.

For each i , there is some $(m_i, R_i, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence $\phi_i: (M_i, x_i) \rightarrow (M_{i+1}, x_{i+1})$, for some $\lambda_i \in (1, e^{r_i})$ and $\varepsilon_i \in (0, r_i)$. Then $\bar{\lambda}_i := \prod_{j \geq i} \lambda_j < e^{\bar{r}_i}$ and $\bar{\varepsilon}_i := \sum_{j \geq i} \varepsilon_j < \bar{r}_i$. Moreover each ϕ_i can be assumed to be C^∞ by Remark 2-(iii). For $i < j$, the pointed local quasi-equivalence $\psi_{ij} = \phi_{j-1} \circ \dots \circ \phi_i: (M_i, f_i, x_i) \rightarrow (M_j, x_j, f_j)$ is of type $(m_i, R_i/\bar{\lambda}_i, \bar{\lambda}_i, \bar{r}_i)$ by Lemma 3.5-(i).

For $i, m \in \mathbb{N}$, let

$$\begin{aligned} B_i &= B_i(x_i, R_i), & B'_i &= B_i(x_i, R'_i), & B''_i &= B_i(x_i, R''_i), \\ B_i^{(m)} &= B_i^{(m)}(x_i, R_i), & B_i'^{(m)} &= B_i^{(m)}(x_i, R'_i), & B_i''^{(m)} &= B_i^{(m)}(x_i, R''_i). \end{aligned}$$

A bar is added to this notation when the corresponding closed balls are considered. We have $\phi_i(\bar{B}_i) \subset B_{i+1}$ because $R_{i+1} > \lambda_i R_i$, and $\phi_i^{(m_i)}(\bar{B}_i''^{(m_i)}) \subset B_{i+1}'^{(m_i)} \subset B_{i+1}''^{(m_i)}$ since $R'_{i+1} > \lambda_i R''_i$ and $g_{i+1}^{(m_i)}$ is the restriction of $g_{i+1}^{(m_i+1)}$. Furthermore $B_i'' \subset \text{dom } \psi_{ij}$ and $B_i''^{(m_i)} \subset \text{dom } \psi_{ij}^{(m_i)}$ for $i < j$ because $R'' \leq R_i/\bar{\lambda}_i$. Therefore $\psi_{ij}(B_i) \subset B_j$ and $\psi_{ij}^{(m_i)}(B_i''^{(m_i)}) \subset B_j^{(m_j)}$. Take compact domains, $\Omega_i \subset M_i$ and $\Omega_i^{(m_i)} \subset T^{(m_i)} M_i$, such that $B_i \subset \Omega_i \subset \text{Int}(\Omega_i^{(m_i)})$ and $B_i'^{(m_i)} \subset \Omega_i^{(m_i)} \subset B_i''^{(m_i)}$; thus $\Omega_i \subset B_i''$ since M_i is a totally geodesic Riemannian submanifold of $T^{(m_i)} M_i$.

According to the proof of [1, Proposition 7.2], there is a pointed complete connected Riemannian manifold (\widehat{M}, \hat{x}) , and, for each i , there is some C^∞ pointed map $\psi_i: (B_i, x_i) \rightarrow (\widehat{M}, \hat{x})$ such that $\psi_{i*}^{(m_i)}: \Omega_i^{(m_i)} \rightarrow T^{(m_i)} \widehat{M}$ is a $\bar{\lambda}_i$ -quasi-isometry, and $\psi_i = \psi_j \circ \psi_{ij}$ for $j \geq i$. Let $\widehat{B}_i = \psi_i(B_i)$, $\widehat{\Omega}_i = \psi_i(\Omega_i)$ and $\widehat{\Omega}_i^{(m_i)} = \psi_{i*}^{(m_i)}(\Omega_i^{(m_i)})$. Let $\hat{f}_i \in C^\infty(\widehat{B}_i, \mathbb{E})$ be determined by $\psi_i^* \hat{f}_i = f_i|_{B_i}$.

Claim 1. For all i , the sequence $\hat{f}_j|_{\widehat{\Omega}_i}$ ($j \geq i$) is convergent in $C^{m_i}(\widehat{\Omega}_i, \mathbb{E})$.

This assertion follows by showing that the restrictions of the functions $f_{ij} := \psi_{ij}^* f_j$ to Ω_i , for $j \geq i$, form a convergent sequence in $C^{m_i}(\Omega_i, \mathbb{E})$. Equivalently, we show that $f_{ij}|_{\Omega_i}$ is a Cauchy sequence with respect to $\|\cdot\|_{C^{m_i}, \Omega_i, g_i}$. For $k \geq j \geq i$,

$$\begin{aligned} \|f_{ij}^{(m_i)} - f_{ik}^{(m_i)}\|_{\Omega_i^{(m_i)}} &= \|f_{j*}^{(m_i)} - f_{k*}^{(m_i)}\|_{\psi_{ij}(\Omega_i^{(m_i)})} \\ &\leq \|f_{j*}^{(m_i)} - f_{jj+1*}^{(m_i)}\|_{\psi_{ij}(\Omega_i^{(m_i)})} + \dots + \|f_{k-1*}^{(m_i)} - f_{k-1k*}^{(m_i)}\|_{\psi_{i, k-1*}^{(m_i)}(\Omega_i^{(m_i)})} \\ &\leq \|f_{j*}^{(m_j)} - f_{jj+1*}^{(m_j)}\|_{\Omega_j^{(m_j)}} + \dots + \|f_{k-1*}^{(m_{k-1})} - f_{k-1k*}^{(m_{k-1})}\|_{\Omega_{k-1}^{(m_{k-1})}} \leq \varepsilon_j + \dots \varepsilon_{k-1} < \bar{\varepsilon}_j \end{aligned} \quad (4)$$

because

$$\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}) \subset \psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)} \subset \Omega_j^{(m_j)}$$

and $f_{jk*}^{(m_j)} = f_{jk*}^{(m_i)}$ on $\Omega_j^{(m_j)} \cap B_j^{(m_i)} \supset \psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)})$.

Let \mathcal{U}_i be a finite collection of charts of M_i with domains $U_{i,a}$, and let $\mathcal{K}_i = \{K_{i,a}\}$ be a family of compact subsets of M_i , with the same index set as \mathcal{U}_i , such that $K_{i,a} \subset U_{i,a}$ for all a , and $\bar{B}_i'' \subset \bigcup_a K_{i,a} =: K_i$.

Thus $\Omega_i \subset K_i$. Choose some $C_i \geq 1$ satisfying (1) for \mathbb{E} -valued functions with \mathcal{U}_i , \mathcal{K}_i , Ω_i and g_i . With the notation of Section 2, for any $\rho > 0$ and $n+1 \leq \mu \leq 2^{m_i}n$, let $\sigma_{i,a,\rho,\mu}^{(m_i)} : U_{i,a} \rightarrow U_{i,a}^{(m_i)}$ be the section of each $\pi : U_{i,a}^{(m_i)} \rightarrow U_{i,a}$ of the type used in Lemma 2.1-(ii). Since $\Omega_i \subset \text{Int}(\Omega_i^{(m_i)})$, there is some $\rho > 0$ so that $\sigma_{i,a,\rho,\mu}^{(m_i)}(K_{i,a} \cap \Omega_i) \subset K_{i,a}^{(m_i)} \cap \Omega_i^{(m_i)}$ for all a and μ . Thus, by Lemma 2.1-(ii), given any $\varepsilon > 0$, there is some $\delta > 0$, depending on ε and ρ , such that

$$\|f_{ij*}^{(m_i)} - f_{ik*}^{(m_i)}\|_{\Omega_i^{(m_i)}} < \delta \implies \|f_{ij} - f_{ik}\|_{C^m, \Omega_i, \mathcal{U}_i, \mathcal{K}_i} < \varepsilon/C_i. \quad (5)$$

For j large enough, we have $\bar{\varepsilon}_j < \delta$, giving

$$\|f_{ij*}^{(m_i)} - f_{ik*}^{(m_i)}\|_{\Omega_i^{(m_i)}} < \delta \implies \|f_{ij} - f_{ik}\|_{C^{m_i}, \Omega_i, \mathcal{U}_i, \mathcal{K}_i} < \varepsilon/C_i \implies \|f_{ij} - f_{ik}\|_{C^{m_i}, \Omega_i, g_i} < \varepsilon$$

by (4), (5) and (1). This shows that $f_{ij}|_{\Omega_i}$ is a Cauchy sequence in the Banach space $C^{m_i}(\Omega_i, \mathbb{E})$ with $\|\cdot\|_{C^{m_i}, \Omega_i, g_i}$, and therefore it is convergent. This completes the proof of Claim 1.

According to Claim 1, for each i , let $\hat{f}_{i\infty} = \lim_{k \rightarrow \infty} \hat{f}_k|_{\hat{\Omega}_i}$ in $C^{m_i}(\hat{\Omega}_i, \mathbb{E})$. Obviously, $\hat{f}_{j\infty}|_{\hat{\Omega}_i} = \hat{f}_{i\infty}$ for $j > i$. Hence there is a function $\hat{f} \in C^\infty(\widehat{M}, \mathbb{E})$ whose restriction to every $\hat{\Omega}_i$ is $\hat{f}_{i\infty}$. From (4), we get $\|\hat{f}_{i*}^{(m_i)} - \hat{f}_{k*}^{(m_i)}\|_{\hat{\Omega}_i^{(m_i)}} < \bar{\varepsilon}_i$ for $k \geq i$, yielding $\|\hat{f}_{i*}^{(m_i)} - \hat{f}_{*}^{(m_i)}\|_{\hat{\Omega}_i^{(m_i)}} \leq \bar{\varepsilon}_i$. Hence $\psi_i : (M_i, f_i, x_i) \mapsto (\widehat{M}, \hat{x}, \hat{f})$ is an $(m_i, R'_i, \bar{\lambda}_i, \bar{\varepsilon}_i)$ -pointed local quasi-equivalence. It follows that $([M_i, f_i, x_i], [\widehat{M}, \hat{x}, \hat{f}]) \in \widehat{U}_{R'_i, s_i}^{m_i}$ for any sequence $s_i \downarrow 0$ so that $s_i > \max\{\ln \bar{\lambda}_i, \bar{\varepsilon}_i\}$, obtaining that $[M_i, f_i, x_i] \rightarrow [\widehat{M}, \hat{x}, \hat{f}]$ as $i \rightarrow \infty$ in $\widehat{\mathcal{M}}_*(n)$. \square

Corollary 4.10. $\widehat{\mathcal{M}}_*(n)$ is Polish.

Proof. This is the content of Propositions 4.8 and 4.9 together. \square

Corollaries 4.7 and 4.10 give Theorem 1.3.

5. FOLIATED STRUCTURE OF $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$

The properties stated in Theorem 1.4 are given by propositions of this section.

Proposition 5.1. $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is Polish.

Proof. For each $R > 0$, let $\mathcal{W}_R \subset \widehat{\mathcal{M}}_*^\infty(n)$ be the open subset consisting of the points $[M, f, x]$ such that $f|_\Omega$ is an immersion for some compact domain $\Omega \subset M$ containing $B_M(x, R)$. Then $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n) = \bigcap_{R=1}^\infty \mathcal{W}_R$ is a G_δ in $\widehat{\mathcal{M}}_*^\infty(n)$. So $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is a Polish space by Corollary 4.10 and [20, Theorem I.3.11]. \square

Proposition 5.2. $\widehat{\mathcal{M}}_{*,\text{imm},c}^\infty(n)$ is dense in $\widehat{\mathcal{M}}_{*,c}^\infty(n)$.

Proof. With the notation of the proof of Proposition 4.8, $\widehat{\mathcal{M}}_{*,c}^\infty(n)$ has an open partition consisting of the subspaces

$$\widehat{\mathcal{M}}_*^\infty(M) = \{[M, f, x] \mid f \in C^\infty(M, \mathbb{E}), x \in M\} \quad (M \in \mathcal{C}).$$

Thus it is enough to prove that each intersection $\widehat{\mathcal{M}}_*^\infty(M) \cap \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is dense in $\widehat{\mathcal{M}}_*^\infty(M)$. This means that $C_{\text{imm}}^\infty(M, \mathbb{E})$ is dense in $C^\infty(M, \mathbb{E})$, which follows easily from [18, Theorem 2.2.12]. \square

Proposition 5.3. There is a connected complete open Riemannian manifold N and some $h \in C_{\text{imm}}^\infty(N, \mathbb{E})$ such that $\hat{\iota}_{N,h}$ is dense in $\widehat{\mathcal{M}}_{*,\text{imm},o}^\infty(n)$.

Proof. In the proof of Proposition 4.8, we can assume that $\mathcal{F}_M \subset C_{\text{imm}}^\infty(M, \mathbb{E})$ for each $M \in \mathcal{C}$ by [18, Theorem 2.2.12]. Then the set (3), denoted here by $\{(M_i, g_i), f_i, x_i\} \mid i \in \mathbb{N}\}$, is contained in $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.

For every i , let $r_i = \max_{x \in M_i} d(x_i, x)$, and let $B_i = B_i(x_i, r_i/2)$ and $B'_i = B_i(x_i, 2r_i/3)$. Let N be a C^∞ connected manifold obtained by modifying $\bigsqcup_i M_i$ on the complement of $\bigsqcup_i \overline{B'_i}$; for instance, we can take N equal to the C^∞ connected sum $M_0 \# M_1 \# \dots$, constructed by removing balls in the sets $M_i \setminus \overline{B'_i}$. Equip N with a complete Riemannian metric g_N whose restriction to each B_i is g_i . For instance, we can take $g_N = \lambda g' + \mu g''$, where $\{\lambda, \mu\}$ is a C^∞ partition of unity of N subordinated to the open covering $\{\bigsqcup_i B'_i, N \setminus \bigsqcup_i \overline{B'_i}\}$, g' is the combination of the metrics g_i on $\bigsqcup_i B'_i$, and g'' is any complete metric on N .

Form [18, Theorems 2.1.1 and 2.2.12], it easily follows that there is some $h \in C_{\text{imm}}^\infty(N, \mathbb{E})$ whose restriction to each B_i is f_i . It is easy to see that N and h satisfies the conditions of the statement. \square

Remark 4. The versions of Propositions 5.2 and 5.3 with embeddings instead of immersions also hold by [18, Theorems 2.1.4 and 2.2.13].

To define foliated charts in $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$, fix some $e \in \mathbb{E}$, and some linear subspace, $V \subset \mathbb{E}$, of dimension n . Let $\Pi_V : \mathbb{E} \rightarrow V$ denote the orthogonal projection. For each complete connected Riemannian manifold M and any $f \in C_{\text{imm}}^\infty(M, \mathbb{E})$, let $\chi_{M,f} = \chi_{V,e,M,f} : M \rightarrow V$ be the C^∞ map defined by $\chi_{M,f}(x) = \Pi_V(f(x) - e)$. Let $\chi = \chi_{V,e} : \mathcal{M}_{*,\text{imm}}^\infty(n) \rightarrow V$ be defined by $\chi([M, f, x]) = \chi_{M,f}(x)$.

Lemma 5.4. χ is continuous

Proof. The map χ equals the following composite of continuous maps:

$$\mathcal{M}_{*,\text{imm}}^\infty(n) \xrightarrow{\text{ev}} \mathbb{E} \xrightarrow{-e} \mathbb{E} \xrightarrow{\Pi_V} V, \quad (6)$$

where the translation by $-e$ in \mathbb{E} is also denoted by $-e$. \square

Given $\rho, \sigma > 0$ and $\kappa > 1$, let $B = B_V(0, \sigma)$, and consider the following subsets of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$:

- $\mathcal{N}_0 = \mathcal{N}_0(V, e, \rho, \kappa, \sigma)$ consists of the classes $[M, f, x] \in \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ such that $\chi_{M,f} : B_M(x, \tilde{\rho}) \rightarrow V$ is a $\tilde{\kappa}$ -quasi-isometric embedding for some $\tilde{\rho} > 5\rho + \kappa\sigma$ and $\tilde{\kappa} \in (1, \kappa)$, and $\overline{B} \subset \chi_{M,f}(B_M(x, \rho))$.
- $\mathcal{N}_1 = \mathcal{N}_1(V, e, \rho, \kappa, \sigma)$ consists of the classes $[M, f, x] \in \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ such that $[M, f, x'] \in \mathcal{N}_0$ for some $x' \in B_M(x, \rho)$.
- $\mathcal{N}_2 = \mathcal{N}_2(V, e, \rho, \kappa, \sigma) := \mathcal{N}_1 \cap \chi^{-1}(B)$.

Using [18, Lemma 2.1.3], it easily follows that, for each $i \in \{0, 1, 2\}$, the sets $\mathcal{N}_i(V, e, \rho, \kappa, \sigma)$ form an open covering of $\mathcal{M}_{*,\text{imm}}^\infty(n)$ by varying $(V, e, \rho, \kappa, \sigma)$.

Lemma 5.5. $\chi_{M,f} : B_M(x, 4\rho + \kappa\sigma) \rightarrow V$ is an embedding for all $[M, f, x] \in \mathcal{N}_1$.

Proof. For each $[M, f, x] \in \mathcal{N}_1$, take some $x' \in B_M(x, \rho)$ so that $[M, f, x'] \in \mathcal{N}_0$. Then $B_M(x, 4\rho + \kappa\sigma) \subset B_M(x', 5\rho + \kappa\sigma)$ and $\chi_{M,f} : B_M(x', 5\rho + \kappa\sigma) \rightarrow V$ is an embedding. \square

Let $\mathcal{Z} = \mathcal{N}_1 \cap \chi^{-1}(0)$, which is closed in \mathcal{N}_2 . For each $[M, f, x] \in \mathcal{N}_1$, there is some $x' \in B_M(x, \rho)$ so that $[M, f, x'] \in \mathcal{N}_0$. Then there is some $x'' \in B_M(x', \rho)$ such that $\chi_{M,f}(x'') = 0$. Observe that $[M, f, x''] \in \mathcal{N}_1$, and therefore $[M, f, x''] \in \mathcal{Z}$. By Lemma 5.5, x'' is the unique point in $B_M(x, 2\rho)$ such that $\chi_{M,f}(x'') = 0$. Thus the class $[M, f, x'']$ depends only on $[M, f, x]$. So a map $\Theta : \mathcal{N}_1 \rightarrow \mathcal{Z}$ is well defined by setting $\Theta([M, f, x]) = [M, f, x'']$.

Lemma 5.6. Θ is continuous.

Proof. Consider a convergent sequence $[M_i, f_i, x_i] \rightarrow [M, f, x]$ in \mathcal{N}_1 . Take points $x'_i \in B_i(x_i, 2\rho)$ and $x' \in B_M(x', 2\rho)$ such that $\chi_{M_i, f_i}(x'_i) = \chi_{M, f}(x') = 0$. Thus $\Theta([M_i, f_i, x_i]) = [M_i, f_i, x'_i]$ and $\Theta([M, f, x]) = [M, f, x']$.

Given $m \in \mathbb{N}$ and $R, r > 0$, for i large enough, there is an $(m, R, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence $\phi_i : (M, f, x) \rightarrow (M_i, f_i, x_i)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Suppose that $R > 3\rho$ and $e^r < 3/2$; in particular, $\overline{B}_M(x, 3\rho) \subset \text{dom } \phi_i$.

Claim 2. $B_i(x_i, 2\rho) \subset \phi_i(B_M(x, 3\rho))$.

The set $A = B_i(x_i, 2\rho) \cap \phi_i(B_M(x, 3\rho))$ contains x_i and is open in the connected space $B_i(x_i, 2\rho)$. Then Claim 2 follows by showing that A is also closed in $B_i(x_i, 2\rho)$. This holds since $A = B_i(x_i, 2\rho) \cap \phi_i(\overline{B}_M(x, 3\rho))$ because, for every $y \in M$ with $d_M(x, y) = 3\rho$, we have

$$d_i(x_i, \phi_i(y)) \geq \frac{1}{\lambda_i} d_M(x, y) > 3\rho e^{-r} > 2\rho.$$

According to Claim 2, there is some $\bar{x}'_i \in B_M(x, 3\rho)$ such that $\phi_i(\bar{x}'_i) = x'_i$. We have

$$\begin{aligned} d_M(x', \bar{x}'_i) &\leq \kappa \|\chi_{M,f}(x') - \chi_{M,f}(\bar{x}'_i)\| = \kappa \|\chi_{M,f}(\bar{x}'_i) - \chi_{M_i, f_i}(x'_i)\| \\ &\leq \kappa \|f(\bar{x}'_i) - f_i(x'_i)\| = \kappa \|f(\bar{x}'_i) - f_i \circ \phi(\bar{x}'_i)\| < \kappa \varepsilon_i < \kappa r. \end{aligned}$$

Therefore, by the continuity of $\hat{l}_{M,f}$, for any $S, s > 0$, if r is small enough and i large enough, there is an (m, S, μ_i, δ_i) -pointed local quasi-equivalence $\psi_i : (M, f, x') \rightarrow (M, f, \bar{x}'_i)$ with $\mu_i \in (1, e^{s/2})$ and $\delta_i \in (0, s/2)$. On the other hand, observe that $\phi_i : (M, \bar{x}'_i, f) \rightarrow (M_i, f_i, x'_i)$ is an $(m, R - 2\rho, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence. Hence, if moreover $R > e^{s/2}S + 2\rho$ and $r < s/2$, we get that $\phi_i \circ \psi_i : (M, f, x') \rightarrow (M_i, f_i, x'_i)$ is an $(m, S, \mu_i \lambda_i, \delta_i + \varepsilon_i)$ -pointed local quasi-equivalence with $\mu_i \lambda_i \in (1, e^s)$ and $\delta_i + \varepsilon_i \in (0, s)$ by Lemma 3.5-(i). This shows that $[M_i, f_i, x'_i] \rightarrow [M, f, x']$ in $\hat{\mathcal{M}}_*^\infty(n)$. \square

Let $\Phi = (\chi, \Theta) : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$.

Lemma 5.7. Φ is bijective, and $\Phi^{-1}(v, [M, f, x]) = [M, f, x']$ for each $(v, [M, f, x]) \in B \times \mathcal{Z}$, where x' is the unique point in $B_M(x, 2\rho) \cap \chi_{M,f}^{-1}(v)$.

Proof. To prove that Φ is injective, let $[M_i, f_i, x_i] \in \mathcal{N}_2$ ($i \in \{1, 2\}$) such that $\Phi([M_1, f_1, x_1]) = \Phi([M_2, f_2, x_2])$; i.e., $\chi_{M_1, f_1}(x_1) = \chi_{M_2, f_2}(x_2)$ and $[M_1, f_1, x'_1] = [M_2, f_2, x'_2]$ for points $x'_i \in B_i(x_i, 2\rho)$ with $\chi_{M_i, f_i}(x'_i) = 0$. Thus there is a pointed equivalence $\phi : (M_1, f_1, x'_1) \rightarrow (M_2, f_2, x'_2)$. We get $\phi(x_1) = x_2$ because $\chi_{M_2, f_2} \circ \phi(x_1) = \chi_{M_1, f_1}(x_1) = \chi_{M_2, f_2}(x_2)$, the map $\chi_{M_i, f_i} : (B_i(x'_i, 2\rho), x'_i) \rightarrow (V, 0)$ is a pointed embedding (Lemma 5.5), and $x_i \in B_i(x'_i, 2\rho)$. So $\phi : (M_1, f_1, x_1) \rightarrow (M_2, f_2, x_2)$ is a pointed equivalence, and therefore $[M_1, f_1, x_1] = [M_2, f_2, x_2]$.

Now, let us prove that Φ is surjective, showing the stated expression of Φ^{-1} . Let $(v, [M, f, x]) \in B \times \mathcal{Z}$. There is some $y \in B_M(x, \rho)$ such that $[M, f, y] \in \mathcal{N}_0$. So there is some $x' \in B_M(y, \rho)$ such that $\chi_{M,f}(x') = v$. It follows that $[M, f, x'] \in \mathcal{N}_1$, $\Theta([M, f, x']) = [M, f, x]$ and $\chi([M, f, x']) = v$. Therefore $[M, f, x'] \in \mathcal{N}_2$ and $\Phi([M, f, x']) = (v, [M, f, x])$. Moreover x' is the unique point in $B_M(x, 2\rho) \cap \chi_{M,f}^{-1}(v)$ by Lemma 5.5. \square

Lemma 5.8. Φ^{-1} is continuous.

Proof. Consider a convergent sequence $(v_i, [M_i, f_i, x_i]) \rightarrow (v, [M, f, x])$ in $B \times \mathcal{Z}$. Take points $x'_i \in B_i(x_i, 2\rho)$ and $x' \in B_M(x, 2\rho)$ such that $\chi_{M_i, f_i}(x'_i) = v_i$ and $\chi_{M,f}(x') = v$. Thus $\Phi^{-1}(v_i, [M_i, f_i, x_i]) = [M_i, f_i, x'_i]$ and $\Phi^{-1}(v, [M, f, x]) = [M, f, x']$.

Given $m \in \mathbb{N}$ and $R, r > 0$, if i is large enough, then $\|v - v_i\| < r$, and there is an $(m, R, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence $\phi_i : (M, f, x) \rightarrow (M_i, f_i, x_i)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Suppose that $R > 3\rho$ and $e^r < 3/2$; in particular, $\bar{B}_M(x, 3\rho) \subset \text{dom } \phi_i$. Like in Claim 2, we get $B_i(x_i, 2\rho) \subset \phi_i(B_M(x, 3\rho))$. Then, since $x'_i \in B_i(x_i, 2\rho)$, there is some $\bar{x}'_i \in B_M(x, 3\rho)$ such that $\phi_i(\bar{x}'_i) = x'_i$. We have

$$\begin{aligned} d_M(x', \bar{x}'_i) &\leq \kappa \|\chi_{M,f}(x') - \chi_{M,f}(\bar{x}'_i)\| \leq \kappa (\|\chi_{M,f}(\bar{x}'_i) - \chi_{M_i, f_i}(x'_i)\| + \|v - v_i\|) \\ &< \kappa (\|f(\bar{x}'_i) - f_i(x'_i)\| + r) = \kappa (\|f(\bar{x}'_i) - f_i \circ \phi(\bar{x}'_i)\| + r) < \kappa(\varepsilon_i + r) < 2\kappa r. \end{aligned}$$

Hence we get $[M_i, f_i, x'_i] \rightarrow [M, f, x']$ in $\hat{\mathcal{M}}_*^\infty(n)$ like in the end of the proof of Lemma 5.6. \square

Corollary 5.9. Φ is a homeomorphism.

Proof. This follows from Lemmas 5.4, 5.6, 5.7 and 5.8. \square

Lemma 5.10. If $[M, f, x] \in \chi^{-1}(B)$ and $[M, f, x'] \in \mathcal{Z}$ for some $x' \in B_M(x, 2\rho)$, then $[M, f, x] \in \mathcal{N}_2$.

Proof. Let $v = \chi([M, f, x]) \in B$. By Lemma 5.7, there is some $x'' \in B_M(x', 2\rho)$ such that $[M, f, x''] \in \mathcal{N}_2$ and $\Phi([M, f, x'']) = (v, [M, f, x'])$. Then $x = x''$ by Lemma 5.5 applied to $\chi_{M,f} : B_M(x', 2\rho) \rightarrow V$. \square

Take $(\tilde{V}, \tilde{e}, \tilde{\rho}, \tilde{\kappa}, \tilde{\sigma})$ like $(V, e, \rho, \kappa, \sigma)$. Let $\tilde{\mathcal{N}}_i = \mathcal{N}_i(\tilde{V}, \tilde{e}, \tilde{\rho}, \tilde{\kappa}, \tilde{\sigma})$ for $i \in \{0, 1, 2\}$, and let $\tilde{\Phi} = (\tilde{\chi}, \tilde{\Theta}) : \tilde{\mathcal{N}}_2 \rightarrow \tilde{B} \times \tilde{\mathcal{Z}}$ be defined like $\Phi = (\chi, \Theta) : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$, using $(\tilde{V}, \tilde{e}, \tilde{\rho}, \tilde{\kappa}, \tilde{\sigma})$. Moreover, for each $[M, f, x] \in \hat{\mathcal{M}}_*^\infty(n)$, let $\tilde{\chi}_{M,f} : M \rightarrow \tilde{V}$ be defined like $\chi_{M,f} : M \rightarrow V$, using $\Pi_{\tilde{V}}$ and \tilde{e} . Suppose that $\mathcal{N}_2 \cap \tilde{\mathcal{N}}_2 \neq \emptyset$, and consider the map $\tilde{\Phi} \circ \Phi^{-1} : \Phi(\mathcal{N}_2 \cap \tilde{\mathcal{N}}_2) \rightarrow \tilde{\Phi}(\mathcal{N}_2 \cap \tilde{\mathcal{N}}_2)$.

Lemma 5.11. Let $(v, [M, f, x]) \in \Phi(\mathcal{N}_2 \cap \tilde{\mathcal{N}}_2)$. Then $\tilde{\Phi} \circ \Phi^{-1}(v, [M, f, x]) = (\tilde{v}, [M, f, \tilde{x}])$, where $\tilde{x} \in \tilde{\chi}_{M,f}^{-1}(0)$ is determined by the condition

$$B_M(x, 2\rho) \cap B_M(\tilde{x}, 2\tilde{\rho}) \cap \chi_{M,f}^{-1}(B) \cap \tilde{\chi}_{M,f}^{-1}(\tilde{B}) \neq \emptyset, \quad (7)$$

and \tilde{v} is the image of v by the composite

$$\chi_{M,f}(O) \xrightarrow{\chi_{M,f}^{-1}} O \xrightarrow{\tilde{\chi}_{M,f}} \tilde{\chi}_{M,f}(O), \quad (8)$$

where $O = B_M(x, 2\rho) \cap B_M(\tilde{x}, 2\tilde{\rho})$.

Proof. Let $[M, f, x'] \in \mathcal{N}_2 \cap \tilde{\mathcal{N}}_2$ such that $\Phi([M, f, x']) = (v, [M, f, x])$ and $\tilde{\Phi}([M, f, x']) = (\tilde{v}, [M, f, \tilde{x}])$. By Lemma 5.7, this means that $\chi_{M,f}(x) = \tilde{\chi}_{M,f}(\tilde{x}) = 0$, $x' \in B_M(x, 2\rho) \cap B_M(\tilde{x}, 2\tilde{\rho})$, $\chi_{M,f}(x') = v$ and $\tilde{\chi}_{M,f}(x') = \tilde{v}$, obtaining (7) and (8). Note that (8) makes sense by Lemma 5.5.

Now, assume that (7) also holds using another point $\tilde{y} \in \tilde{\chi}_{M,f}^{-1}(0)$ instead of \tilde{x} . Thus there is some $y' \in B_M(x, 2\rho) \cap B_M(\tilde{x}, 2\tilde{\rho})$ with $w := \chi_{M,f}(y') \in B$ and $\tilde{w} := \tilde{\chi}_{M,f}(y') = \tilde{B}$. Then $[M, f, y'] \in \mathcal{N}_2$ by Lemma 5.10, and $\Phi([M, f, y']) = (w, [M, f, x])$ and $\tilde{\Phi}([M, f, y']) = (w, [M, f, \tilde{y}])$. We have

$$d_M(\tilde{x}, \tilde{y}) \leq d_M(\tilde{x}, x') + d_M(x', y') + d_M(y', \tilde{y}) < 4\tilde{\rho} + \tilde{\kappa}\|\tilde{v} - \tilde{w}\| < 4\tilde{\rho} + \tilde{\kappa}\tilde{\sigma}.$$

Since moreover $\tilde{\chi}_{M,f}(\tilde{x}) = 0 = \tilde{\chi}_{M,f}(\tilde{y})$, we get $\tilde{x} = \tilde{y}$ by Lemma 5.5. \square

Proposition 5.12. *All possible maps $\Phi : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ form an atlas of a C^∞ foliated structure on $\hat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.*

Proof. The maps $\Phi : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ are homeomorphisms (Corollary 5.9). All possible sets \mathcal{N}_2 form an open cover of $\hat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$. Moreover, in Lemma 5.11, it follows from (7) that $[M, f, \tilde{x}]$ depends only on $[M, f, x]$. Thus all possible maps $\Phi : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ form an atlas of a foliated structure on $\hat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.

With the notation of Lemma 5.11 and the terminology of [1, Section 2.1], it only remains to show that $\tilde{\Phi} \circ \Phi^{-1}$ is C^∞ ; i.e., to prove that the mapping $(v, [M, f, x]) \mapsto \tilde{v}$ is C^∞ . First, note that, for each fixed $[M, f, x]$, the mapping $v \mapsto \tilde{v}$ is C^∞ because (8) is C^∞ . Consider now a convergent sequence $[M_i, f_i, x_i] \rightarrow [M, f, x]$ in \mathcal{Z} . Let $\tilde{x}_i \in \tilde{\chi}_{M_i, f_i}^{-1}(0)$ be determined by

$$B_M(x_i, 2\rho) \cap B_M(\tilde{x}_i, 2\tilde{\rho}) \cap \chi_{M_i, f_i}^{-1}(B) \cap \tilde{\chi}_{M_i, f_i}^{-1}(\tilde{B}) \neq \emptyset,$$

and let $O_i = B_M(x_i, 2\rho) \cap B_M(\tilde{x}_i, 2\tilde{\rho})$. Given $m \in \mathbb{N}$ and $R, r > 0$, for each i large enough, there is an $(m, R, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence $\phi_i : (M_i, f_i, x_i) \rightarrow (M, f, x)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Let $\Omega_i^{(m)}$ be a compact domain in $\text{dom } \phi_{i*}^{(m)}$ such that $B_i^{(m)}(x_i, R) \subset \Omega_i^{(m)}$ and $\phi_{i*}^{(m)} : (\Omega_i^{(m)}, f_{i*}^{(m)}) \rightarrow (T^{(m)}M, f_*^{(m)})$ is an $(\varepsilon_i, \lambda_i)$ -quasi-equivalence. Since $(\Pi_V)_*^{(m)} \equiv \Pi_{V^{2m}} : T^{(m)}\mathbb{E} \equiv \mathbb{E}^m \rightarrow T^{(m)}V \equiv V^{2m}$, we have

$$\begin{aligned} \left\| \chi_{M_i, f_{i*}}^{(m)} - (\chi_{M, f} \circ \phi_i)_*^{(m)} \right\|_{\Omega_i^{(m)}} &\leq \left\| (f_i - e)_*^{(m)} - ((f - e) \circ \phi_i)_*^{(m)} \right\|_{\Omega_i^{(m)}} \\ &= \left\| f_{i*}^{(m)} - (\phi_i^* f)_*^{(m)} \right\|_{\Omega_i^{(m)}} < \varepsilon_i < r. \end{aligned} \quad (9)$$

Assume that $R > 2e^r\rho$. Then $B_i(x_i, 2\rho) \subset B_i(x_i, R)$, and, like in Claim 2, we also get $B_M(x, 2\rho) \subset \phi_i(B_i(x_i, R))$. Thus $O_i \subset B_i(x_i, R)$ and $O \subset \phi_i(B_i(x_i, R))$. Let $\Xi \subset \chi_{M, f}(O)$ be a compact domain, which is also contained in $\chi_{M_i, f_i}(O_i)$ for i large enough. Let $\Xi^{(m)}$ be a compact domain contained in $T^{(m)}\mathbb{E}$ such that

$$\Xi \subset \text{Int}(\Xi^{(m)}), \quad (\chi_{M_i, f_i}^{-1})_*^{(m)}(\Xi^{(m)}) \subset \Omega_i^{(m)} \cap T^{(m)}O_i, \quad (\chi_{M, f}^{-1})_*^{(m)}(\Xi^{(m)}) \subset \phi_{i*}^{(m)}(\Omega_i^{(m)}) \cap T^{(m)}O.$$

Since the restrictions of $(\chi_{M, f}^{-1})_*^{(m)}$ and $(\tilde{\chi}_{M, f})_*^{(m)}$ to the respective compact domains $\Xi^{(m)}$ and $\chi_{M, f}^{-1}(\Xi^{(m)}) \cap T^{(m)}O$ are C^∞ embeddings, these restrictions are ν -quasi-isometric for some $\nu \geq 1$. Hence, by (9),

$$\begin{aligned} d_i^{(m)} \left((\phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi), (\chi_{M, f}^{-1})_*^{(m)}(\xi) \right) &\leq \nu \left\| (\chi_{M, f} \circ \phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) - \xi \right\| \\ &= \nu \left\| (\chi_{M, f} \circ \phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) - (\chi_{M_i, f_i} \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) \right\| < \nu r, \end{aligned} \quad (10)$$

for all $\xi \in \Xi^{(m)}$. On the other hand, like in (9), we get

$$\left\| \tilde{\chi}_{M_i, f_{i*}}^{(m)} - (\tilde{\chi}_{M, f} \circ \phi_i)_*^{(m)} \right\|_{\Omega_i^{(m)}} < r. \quad (11)$$

Combining (10) and (11), we obtain the following for all $\xi \in \Xi^{(m)}$:

$$\begin{aligned}
& \left\| (\tilde{\chi}_{M_i, f_i} \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) - (\tilde{\chi}_{M, f} \circ \chi_{M, f}^{-1})_*^{(m)}(\xi) \right\| \\
& \leq \left\| (\tilde{\chi}_{M_i, f_i} \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) - (\tilde{\chi}_{M, f} \circ \phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) \right\| \\
& \quad + \left\| (\tilde{\chi}_{M, f} \circ \phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi) - (\tilde{\chi}_{M, f} \circ \chi_{M, f}^{-1})_*^{(m)}(\xi) \right\| \\
& < r + \nu d_i^{(m)} \left((\phi_i \circ \chi_{M_i, f_i}^{-1})_*^{(m)}(\xi), (\chi_{M, f}^{-1})_*^{(m)}(\xi) \right) < (1 + \nu^2)r.
\end{aligned}$$

Note that the same choices of Ξ and $\Xi^{(m)}$ are valid for all r small enough, obtaining that $(\tilde{\chi}_{M_i, f_i} \circ \chi_{M_i, f_i}^{-1})_*^{(m)} \rightarrow (\tilde{\chi}_{M, f} \circ \chi_{M, f}^{-1})_*^{(m)}$ uniformly on $\Xi^{(m)}$. Moreover the same choice of Ξ is valid for all m , and therefore $\tilde{\chi}_{M_i, f_i} \circ \chi_{M_i, f_i}^{-1} \rightarrow \tilde{\chi}_{M, f} \circ \chi_{M, f}^{-1}$ on Ξ with respect to the C^∞ topology by the obvious version of Lemma 2.1 for maps between open subsets of \mathbb{R}^n . Since every point in $\chi_{M, f}(O)$ belongs to some domain Ξ as above if r is chosen small enough, it follows that $\tilde{\Phi} \circ \Phi^{-1}$ is C^∞ . \square

Now, let $\hat{\mathcal{F}}_{*, \text{imm}}^\infty(n)$ denote the C^∞ foliated structure on $\hat{\mathcal{M}}_{*, \text{imm}}^\infty(n)$ defined by the maps Φ according to Proposition 5.12.

Proposition 5.13. *The following properties hold:*

- (i) $\hat{\mathcal{F}}_{*, \text{imm}}^\infty(n)$ is the unique C^∞ foliated structure on $\hat{\mathcal{M}}_{*, \text{imm}}^\infty(n)$ such that its underlying topological foliated structure is $\hat{\mathcal{F}}_{*, \text{imm}}(n)$ and $\text{ev} : \hat{\mathcal{M}}_{*, \text{imm}}^\infty(n) \rightarrow \mathbb{E}$ is a C^∞ immersion.
- (ii) For each $[M, f, x] \in \hat{\mathcal{M}}_{*, \text{imm}}^\infty(n)$, the map $\hat{\iota}_{M, f} : M \rightarrow \text{im } \hat{\iota}_{M, f}$ is a local diffeomorphism, where the leaf $\text{im } \hat{\iota}_{M, f}$ is equipped with the C^∞ structure induced by $\hat{\mathcal{F}}_{*, \text{imm}}^\infty(n)$.

Proof. Take a foliated chart $\Phi : \mathcal{N}_2 \rightarrow B \times \mathbb{Z}$ as above. For each $[M, f, x] \in \mathbb{Z}$, the restriction of $\text{ev} \circ \Phi^{-1}$ to $B \times \{[M, f, x]\} \equiv B$ is the composite

$$B \xrightarrow{\chi_{M, f}^{-1}} B_M(x, 2\rho) \cap \chi_{M, f}^{-1}(B) \xrightarrow{f} \mathbb{E},$$

where the first map is a C^∞ diffeomorphism, and the second one is a C^∞ immersion. Take a convergent sequence $[M_i, f_i, x_i] \rightarrow [M, f, x]$ in \mathbb{Z} , and let $\Xi \subset B$ be any compact domain. Given $R > 2\rho$ and a compact domain $\Omega \subset M$ containing $B_M(x, R)$, there is a C^∞ pointed embedding $\phi_i : (\Omega, x) \rightarrow (M_i, x_i)$ for i large enough such that $\phi_i^* g_i \rightarrow g_M$ and $\phi_i^* f_i \rightarrow f$ on Ω with respect to the C^∞ topology. So $B_i(x_i, R) \subset \phi_i(\Omega)$ for i large enough. Thus also $\phi_i^* \chi_{M_i, f_i} \rightarrow \chi_{M, f}$ on Ω with respect to the C^∞ topology, and therefore $\phi_i^{-1} \circ \chi_{M_i, f_i}^{-1} \rightarrow \chi_{M, f}^{-1}$ on Ξ with respect to the C^∞ topology [18, p. 64, Exercise 9]. Hence

$$f_i \circ \chi_{M_i, f_i}^{-1} - f \circ \chi_{M, f}^{-1} = f_i \circ \phi_i \circ (\phi_i^{-1} \circ \chi_{M_i, f_i}^{-1} - \chi_{M, f}^{-1}) + (f_i \circ \phi_i - f) \circ \chi_{M, f}^{-1} \rightarrow 0$$

on Ξ with respect to the C^∞ topology. Since any element of B is contained some Ξ as above, it follows that $\text{ev} \circ \Phi^{-1}$ is a C^∞ immersion, and therefore $\text{ev} : \hat{\mathcal{M}}_{*, \text{imm}}^\infty(n) \rightarrow \mathbb{E}$ is C^∞ with respect to $\hat{\mathcal{M}}_{*, \text{imm}}^\infty(n)$. This shows (i), except uniqueness.

According to Lemma 5.7, for each chart $\Phi : \mathcal{N}_2 \rightarrow B \times \mathbb{Z}$, the plaque that corresponds to each $[M, f, x] \in \mathbb{Z}$ is $\hat{\iota}_{M, f}(B_M(x, 2\rho) \cap \chi_{M, f}^{-1}(B))$. Moreover the composite

$$B_M(x, 2\rho) \cap \chi_{M, f}^{-1}(B) \xrightarrow{\hat{\iota}_{M, f}} \hat{\iota}_{M, f}(B_M(x, 2\rho) \cap \chi_{M, f}^{-1}(B)) \xrightarrow{\chi} B$$

is the diffeomorphism $\chi_{M, f} : B_M(x, 2\rho) \cap \chi_{M, f}^{-1}(B) \rightarrow B$. This shows that the leaf topology on $\hat{\mathcal{M}}_{*, \text{imm}}^\infty(n)$ equals the topological sum of all possible spaces $\text{im } \hat{\iota}_{M, f}$ with the topology so that $\hat{\iota}_{M, f} : M \rightarrow \text{im } \hat{\iota}_{M, f}$ is a local homeomorphism, obtaining that these spaces are the leaves because they are connected. It also follows that $\hat{\iota}_{M, f} : M \rightarrow \text{im } \hat{\iota}_{M, f}$ is a local diffeomorphism for each leaf $\text{im } \hat{\iota}_{M, f}$. This shows (ii).

Now, suppose $\text{ev} : \hat{\mathcal{M}}_{*, \text{imm}}^\infty(n) \rightarrow \mathbb{E}$ is C^∞ with respect to some C^∞ foliated structure \mathcal{G} whose underlying topological foliated structure is $\hat{\mathcal{F}}_{*, \text{imm}}(n)$. Then $\chi : \hat{\mathcal{M}}_{*, \text{imm}}^\infty(n) \rightarrow V$ is also C^∞ with respect to \mathcal{G} because it equals the composite (6). So each chart $\Phi = (\chi, \Theta) : \mathcal{N}_2 \rightarrow B \times \mathbb{Z}$ is also C^∞ with respect to \mathcal{G} and the C^∞ product foliated structure of $B \times \mathbb{Z}$. Moreover, for all complete connected Riemannian manifold

M and $f \in C_{\text{imm}}^\infty(M)$, the map $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ is a C^∞ local diffeomorphism with respect to the C^∞ structure induced by \mathcal{G} on the leaf $\text{im } \hat{\iota}_{M,f}$ because ev is a C^∞ immersion and $\text{ev} \circ \hat{\iota}_{M,f} = f$, which is a C^∞ local embedding. Thus the restriction of $\chi : \mathcal{N}_2 \rightarrow B$ to each plaque is a C^∞ diffeomorphism. Using again [18, p. 64, Exercise 9], it follows that $\Phi : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ is also C^∞ foliated diffeomorphism with respect to the restriction of \mathcal{G} and the C^∞ product foliated structure of $B \times \mathcal{Z}$. This shows that $\mathcal{G} = \hat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$, completing the proof of (i). \square

Consider a leaf $\text{im } \hat{\iota}_{M,f}$ of $\hat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$. Every $x \in M$ has an open neighborhood U in M so that $f : U \rightarrow \mathbb{E}$ is an embedding, obtaining that $\phi(U) \cap U = \emptyset$ for all $\phi \in \text{Iso}(M, f) \setminus \{\text{id}_M\}$. Therefore the subgroup $\text{Iso}(M, f) \subset \text{Iso}(M)$ is discrete, the quotient projection $M \rightarrow \text{Iso}(M, f) \backslash M$ is a covering map, and there is a unique Riemannian structure on the manifold $\text{Iso}(M, f) \backslash M$ so that $M \rightarrow \text{Iso}(M, f) \backslash M$ is a local isometry. Moreover $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ induces a diffeomorphism $\bar{\iota}_{M,f} : \text{Iso}(M, f) \backslash M \rightarrow \text{im } \hat{\iota}_{M,f}$. Thus $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ is a covering map, and $\text{im } \hat{\iota}_{M,f}$ has a unique Riemannian metric so that $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ is a local isometry, and therefore $\bar{\iota}_{M,f} : \text{Iso}(M, f) \backslash M \rightarrow \text{im } \hat{\iota}_{M,f}$ becomes an isometry.

Proposition 5.14. *The above Riemannian metrics on the leaves of $\hat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$ form a C^∞ Riemannian metric on $(\hat{\mathcal{M}}_{*,\text{imm}}^\infty(n), \hat{\mathcal{F}}_{*,\text{imm}}^\infty(n))$.*

Proof. Let $\Phi = (\chi, \Theta) : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ be defined by any choice of $(V, e, \rho, \kappa, \sigma)$ as above, and let $[M_i, f_i, x_i] \rightarrow [M, f, x]$ be a convergent sequence in \mathcal{Z} . Let \bar{g}_M and \bar{g}_i be the metrics on B that correspond to g_M and g_i by the diffeomorphisms

$$\chi_{M,f} : P := B_M(x, 2\rho) \cap \chi_{M,f}^{-1}(B) \rightarrow B, \quad \chi_{M_i,f_i} : P_i := B_i(x_i, 2\rho) \cap \chi_{M_i,f_i}^{-1}(B) \rightarrow B,$$

respectively (see Lemma 5.7). According to the proof of Proposition 5.13-(ii), we have to prove that $\bar{g}_i \rightarrow \bar{g}_M$ as $i \rightarrow \infty$ in the weak C^∞ topology.

Given $m \in \mathbb{N}$, $R, r > 0$, for each i large enough, there is an $(m, R, \lambda_i, \varepsilon_i)$ -pointed local quasi-equivalence $\phi_i : (M, f, x) \rightarrow (M_i, f_i, x_i)$ for some $\lambda_i \in (1, e^r)$ and $\varepsilon_i \in (0, r)$. Assuming $R > 2e^r\rho$, we get $B_M(x, 2\rho) \subset B_M(x, R)$ and $B_i(x_i, 2\rho) \subset \phi_i(B_M(x, R))$, like in the proof of Proposition 5.12. Take a compact domain $\Omega_i^{(m)} \subset \text{dom } \phi_{i*}^{(m)}$ such that $B_i^{(m)}(x_i, R) \subset \Omega_i^{(m)}$ and $\phi_{i*}^{(m)} : \Omega_i^{(m)} \rightarrow T^{(m)}M$ is a $(\lambda_i, \varepsilon_i)$ -quasi-isometry. Let $\Xi \subset B$ be a compact domain, and let $\Xi^{(m)}$ be a compact domain contained in $T^{(m)}B$ such that

$$\Xi \subset \text{Int}(\Xi^{(m)}), \quad (\chi_{M,f}^{-1})_*^{(m)}(\Xi^{(m)}) \cap T^{(m)}P \subset \Omega_i^{(m)}, \quad (\chi_{M_i,f_i}^{-1})_*^{(m)}(\Xi^{(m)}) \cap T^{(m)}P_i \subset \phi_{i*}^{(m)}(\Omega_i^{(m)}).$$

Like in (10), there is some $\nu \geq 1$, independent of i , such that

$$d_i^{(m)} \left((\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})_*^{(m)}(\xi), (\chi_{M,f}^{-1})_*^{(m)}(\xi) \right) < \nu r,$$

for all $\xi \in \Xi^{(m)}$. Since the choice of $\Xi^{(m)}$ is valid for all r small enough, it follows that $\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1} \rightarrow \chi_{M,f}^{-1}$ in $C^m(\Xi, M)$ by the obvious version of Lemma 2.1 for maps between manifolds. Since the choice of Ξ is valid for all m , it follows that this convergence also holds in $C^\infty(\Xi, M)$. Take a compact domain $\Omega \subset M$ such that $B_M(x, R) \subset \Omega$ and $\phi_i^* g_i \rightarrow g_M$ on Ω with respect to the C^∞ topology. We get

$$(\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^*(\phi_i^* g_i - g_M) \rightarrow (\chi_{M,f}^{-1})^* 0 = 0$$

on Ξ with respect to the C^∞ topology. So

$$\begin{aligned} \bar{g}_i - \bar{g}_M &= (\chi_{M_i,f_i}^{-1})^* g_i - (\chi_{M,f}^{-1})^* g_M \\ &= (\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^*(\phi_i^* g_i - g_M) + (\phi_i^{-1} \circ \chi_{M_i,f_i}^{-1})^* g_M - (\chi_{M,f}^{-1})^* g_M \rightarrow 0 \end{aligned}$$

on Ξ with respect to the C^∞ topology. Since every point in B belongs to some domain Ξ as above if r is chosen small enough, it follows that $\bar{g}_i - \bar{g}_M \rightarrow 0$ on B with respect to the weak C^∞ topology. \square

Proposition 5.15. *The holonomy covering of any leaf $\text{im } \hat{\iota}_{M,f}$ of $\hat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$ is $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$.*

This proposition follows directly from the obvious version of [1, Lemma 11.9] for $\hat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.

6. UNIVERSALITY

Definition 6.1. Let X be a sequential Riemannian foliated space with complete leaves, and let L_x denote the leaf through every $x \in X$, whose holonomy covering is denoted by \tilde{L}_x^{hol} . It is said that X is *covering-continuous* when there is a connected pointed covering (\tilde{L}_x, \tilde{x}) of (L_x, x) for all $x \in X$ such that $[\tilde{L}_{x_i}, \tilde{x}_i]$ is C^∞ convergent to $[\tilde{L}_x, \tilde{x}]$ if $x_i \rightarrow x$ is a convergent sequence in X . When this condition is satisfied with $\tilde{L}_x = \tilde{L}_x^{\text{hol}}$ for all $x \in X$, it is said that X is *holonomy-continuous*.

Remark 5. Observe the following:

- (i) Covering-continuity and holonomy-continuity are weaker than covering-determination and holonomy-determination [1, Definition 12.1], which were defined by using “if and only if” instead of “if”.
- (ii) The condition of being covering-continuous is hereditary (by saturated subspaces).
- (iii) Covering/holonomy-continuity/determination have obvious generalizations to arbitrary Riemannian foliated spaces by using nets instead of sequences.

Example 6.2. The following simple examples clarify Definition 6.1:

- (i) The Reeb foliation on S^3 with the standard metric is covering-continuous, but it is not holonomy-continuous with any Riemannian metric. If the metric is modified around the compact leaf $T^2 = S^1 \times S^1$ so that the diffeomorphism $(x, y) \mapsto (y, x)$ of T^2 is not an isometry, then this foliation becomes non-covering-continuous.
- (ii) The Riemannian foliated space of [22, Example 2.5] is covering-determined but not holonomy-continuous. This example can be easily realized as a saturated subspace of a Riemannian foliated space where the holonomy coverings of the leaves are isometric to \mathbb{R} . So holonomy-continuity is not hereditary.
- (iii) $\hat{M}_{*, \text{imm}}^\infty(n)$ is holonomy-continuous. However it is not holonomy-determined for $n \geq 1$ by [1, Remark 10-(iii)], since there are different points with isometric pointed holonomy covers of the corresponding pointed leaves. To see this, take any connected complete Riemannian n -manifold M , and some $x \in M$ and $f, f' \in C_{\text{imm}}^\infty(M, \mathbb{E})$ such that $f(x) \neq f'(x)$. Then $\hat{\iota}_{M, f}(x) \neq \hat{\iota}_{M, f'}(x)$, but (M, x) is isometric to the holonomy covers of the pointed leaves $(\text{im } \hat{\iota}_{M, f}, \hat{\iota}_{M, f}(x))$ and $(\text{im } \hat{\iota}_{M, f'}, \hat{\iota}_{M, f'}(x))$.

Proposition 6.3 (Cf. [4, Theorem 11.4.4]). *For any Polish C^∞ foliated space X with complete leaves, there is a C^∞ embedding $X \rightarrow \mathbb{E}$.*

Proof. This is an adaptation of the usual argument to show the existence of C^∞ embeddings of C^∞ manifolds in Euclidean spaces [18, Theorem 1.3.4]. Let $n = \dim X$ (as foliated space), and let $B_r = B_{\mathbb{R}^n}(0, r)$ and $\overline{B}_r = \overline{B}_{\mathbb{R}^n}(0, r)$ for each $r > 0$.

Claim 3. Let Z be a Polish space, and consider the C^∞ foliated structure on $U := B_2 \times Z$ with leaves $B_2 \times \{*\}$. Let V and W be open subsets of U such that $\overline{V} \subset W$ and $\overline{W} \subset B_1 \times Z$. Then there is some $h \in C^\infty(U)$ such that $h = 1$ on V and $\text{supp } h \subset W$.

Since \overline{B}_1 is compact, it easily follows that each $z \in Z$ has an open neighborhood P_z in Z such that, for some open subsets $G_z, H_z \subset B_2$ with $\overline{G_z} \subset H_z$ and $\overline{H_z} \subset B_1$, we have $\overline{V} \cap (B_1 \times P_z) \subset G_z \times P_z$ and $\overline{H_z} \times P_z \subset W$. Let $\{\lambda_i\}$ be a partition of unity of Z subordinated to the open cover $\{P_z \mid z \in Z\}$; in particular, for every i , there is some $z_i \in Z$ so that $\text{supp } \lambda_i \subset P_{z_i}$. Let $h_i \in C^\infty(B_2)$ such that $h_i = 1$ on G_{z_i} and $\text{supp } h_i \subset H_{z_i}$. Then $h_i \lambda_i \in C^\infty(U)$, $h_i \lambda_i = \lambda_i$ on $G_{z_i} \times P_{z_i}$ and $\text{supp}(h_i \lambda_i) \subset H_{z_i} \times P_{z_i}$. It follows that $h = \sum_i h_i \lambda_i$ satisfies the properties stated in Claim 3.

Now, let \mathcal{U} be a countable collection of C^∞ foliated charts $\phi_i : U_{2,i} \rightarrow B_2 \times Z_i$ of X such that the open sets $U_{1,i} := \phi_i^{-1}(B_1 \times Z_i)$ cover X . Using the paracompactness and regularity of X , a standard argument gives locally finite open covers, $\mathcal{V} = \{V_i\}$ and $\mathcal{W} = \{W_i\}$, with the same index set as \mathcal{U} , such that $\overline{V_i} \subset W_i$ and $\overline{W_i} \subset U_{1,i}$. For each i , let \mathbb{E}_i be a copy of \mathbb{E} . Take embeddings $\psi_i : Z_i \rightarrow \mathbb{E}_i$ [9, Corollary IX.9.2]. Thus each composite

$$U_{2,i} \xrightarrow{\phi_i} B_2 \times Z_i \xrightarrow{\text{id} \times \psi_i} B_2 \times \mathbb{E}_i \hookrightarrow \mathbb{R}^n \times \mathbb{E}_i =: \tilde{\mathbb{E}}_i$$

is a C^∞ embedding with respect to the restriction of \mathcal{F} , which will be denoted by $\tilde{\phi}_i$. By Claim 3, there are functions $h_i \in C^\infty(U_{2,i})$ such that $h_i = 1$ on V_i and $\text{supp } h_i \subset W_i$. Then a C^∞ embedding⁹ $f : X \rightarrow \bigoplus_i \tilde{\mathbb{E}}_i \cong \mathbb{E}$ is defined by $f(x) = \sum_a h_a(x) \tilde{\phi}_{i_{k_a}}$.

□

Proof of Theorem 1.5. The Polish Riemannian foliated space $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ has complete leaves and is holonomy-continuous (Example 6.2-(iii)). Thus any Polish Riemannian foliated subspace of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ is also covering-continuous (Remark 5-(ii)).

Let X be any covering continuous Polish Riemannian foliated space with complete leaves. By Proposition 6.3, there is a C^∞ embedding $f : X \rightarrow \mathbb{E}$. With the notation of Definition 6.1, suppose that the covering-continuity of X is satisfied with the connected pointed coverings $(\tilde{L}_x, \tilde{x}) \rightarrow (L_x, x)$ ($x \in X$). Let $\hat{\iota}_{X,f} : X \rightarrow \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$ be defined by $\hat{\iota}_{X,f}(x) = [\tilde{L}_x, \tilde{f}_x, \tilde{x}]$, where \tilde{f}_x is the lift of $f|_{L_x}$ to \tilde{L}_x . This map is well defined because the leaves of X are complete. Moreover it is obviously foliated and continuous by the definitions of covering-continuity and the topology of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.

To show that $\hat{\iota}_{X,f}$ is C^∞ , take a foliated chart $\Phi = (\chi, \Theta) : \mathcal{N}_2 \rightarrow B \times \mathcal{Z}$ of $\widehat{\mathcal{F}}_{*,\text{imm}}^\infty(n)$ defined by any choice of $(V, e, \rho, \kappa, \sigma)$ as above. Let U be the domain of a foliated chart of X such that $\hat{\iota}_{X,f}(U) \subset \mathcal{N}_2$. Then the composite

$$U \xrightarrow{\hat{\iota}_{X,f}} \mathcal{N}_2 \xrightarrow{\chi} B$$

is equal to $\Pi_V \circ (f - e)$, and therefore it is C^∞ .

Finally, $\hat{\iota}_{X,f}$ is a C^∞ embedding because the composite

$$X \xrightarrow{\hat{\iota}_{X,f}} \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n) \xrightarrow{\text{ev}} \mathbb{E}$$

equals the C^∞ embedding f .

□

7. REALIZATION OF MANIFOLDS OF BOUNDED GEOMETRY AS LEAVES

Proposition 7.1. *Let M be any connected, complete Riemannian n -manifold of bounded geometry. Then there is a C^∞ embedding $f : M \rightarrow \mathbb{E}$ such that $\widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$ is a compact subspace of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.*

Proof. Let $B_r = B_{\mathbb{R}^n}(0, r)$ for each $r > 0$. By the bounded geometry of M , there is some $r > 0$, smaller than the injectivity radius of M , such that the following properties hold:

- (i) For the normal parametrizations $\kappa_x : B_r \rightarrow B_M(x, r)$ ($x \in M$), the corresponding metric coefficients, g_{ij} and g^{ij} , as a family of C^∞ functions on B_r parametrized by x, i and j , lie in a bounded subset of the Fréchet space $C_b^\infty(B_r)$ [28, Theorem A.1], [29, Theorem 2.5] (see also [26, Proposition 2.4], [10]).
- (ii) There is some countable subset $\{x_i \mid i \in \mathbb{N}\} \subset M$ and some $c \in \mathbb{N}$ such that the family of balls $B_M(x_i, r/2)$ covers M , and $B_M(x, r)$ meets at most c sets $B_M(x_i, r)$ for all $x \in M$ [33, A1.2 and A1.3], [29, Proposition 3.2].

Let $\kappa_i = \kappa_{x_i}$ for each i .

Claim 4. There is a partition of \mathbb{N} into finitely many sets, I_1, \dots, I_{c+1} , such that $B_M(x_i, r) \cap B_M(x_j, r) = \emptyset$ for $i \in I_k$ and $j \in I_l$ with $k \neq l$.

This claim follows by considering the graph G whose set of vertices is \mathbb{N} , and such that there is a unique edge connecting two different vertices, i and j , if and only if $B_M(x_i, r) \cap B_M(x_j, r) \neq \emptyset$. Since there are at most c edges meeting at each vertex according to (ii), G is $c+1$ -colorable¹⁰; i.e., there is a partition of \mathbb{N} into subsets, I_1, \dots, I_{c+1} , such that there is no edge joining any pair of different vertices in any I_k .

Let S be an isometric copy in \mathbb{R}^{n+1} of the standard n -dimensional sphere containing the origin 0. Choose some spherically symmetric C^∞ function $\rho \in C^\infty(\mathbb{R}^n)$ such that $\rho(x) = 1$ if $|x| \leq r/2$ and $\rho(x) = 0$ if $|x| \geq r$. Take also some C^∞ map $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ that restricts to a diffeomorphism $B_r \rightarrow S \setminus \{0\}$ and maps $\mathbb{R}^n \setminus B_r$

⁹The notation $\bigoplus_i \mathbb{F}_i$ is used for the Hilbert space direct sum of a family of Hilbert spaces \mathbb{F}_i ; i.e., the Hilbert space completion of $\bigoplus_i \mathbb{F}_i$ with the scalar product $\langle (v_i), (w_i) \rangle = \sum_i \langle v_i, w_i \rangle$.

¹⁰This easily follows by induction, assigning to each i a color different from the colors of the previous vertices that are neighbors of i , which is possible because there are at most c of them (see [3]).

to 0. Let $\tilde{\rho}_i$ be the extension by zero of $\rho \circ \kappa_i^{-1}$ to the whole of M , and let $\tilde{\rho}^k = \sum_{i \in I^k} \tilde{\rho}_i$. For each k , define $f^k : M \rightarrow \mathbb{R}^{n+2}$ by

$$f^k(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{i \in I^k} B_M(x_i, r) \\ (\tilde{\rho}^k(x)/i, \tilde{\rho}^k(x) \cdot \tau \circ \kappa_i^{-1}(x)) & \text{if } x \in B_M(x_i, r) \text{ for some } i \in I^k. \end{cases}$$

So $f^k \circ \kappa_i = (\rho/i, \rho \cdot \tau)$, obtaining that, for every multi-index α , the function $|\partial_\alpha(f^k \circ \kappa_i)|$ is uniformly bounded over B_r by a constant depending only on $|\alpha|$. Let $f = (f^1, \dots, f^{c+1}) : M \rightarrow \mathbb{R}^{(c+1)(n+2)}$. We have $\sup_M |\nabla^m f| < \infty$ for each $m \in \mathbb{N}$ by (i). Moreover $f^k \circ \kappa_i = (1/i, \tau)$ on $B_{r/2}$, obtaining that f is a C^∞ embedding, and $\inf_M |\bigwedge^n df| > 0$ by (i). By taking any isometric linear embedding of $\mathbb{R}^{(c+1)(n+2)}$ into \mathbb{E} , we can consider $\mathbb{R}^{(c+1)(n+2)}$ -valued functions as \mathbb{E} -valued functions; in particular, this applies to f .

Claim 5. $\widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f}) \subset \widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$.

This claim is true because, for all $[N, h, y] \in \widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$, it is easy to see that $\inf_N |\bigwedge^n dh| \geq \inf_M |\bigwedge^n df| > 0$, obtaining that h is an immersion.

Claim 6. $\widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$ is compact.

This assertion follows by showing that any sequence in $\text{im } \hat{\iota}_{M,f}$ has a subsequence that is convergent in $\widehat{\mathcal{M}}_*^\infty(n)$. Assume first that the sequence is of the form $[M, f, x_{i_p}]$ for some sequence of indices i_p . Since $\widehat{\text{Cl}}_\infty(\text{im } \iota_M)$ is compact in $\mathcal{M}_*^\infty(n)$ by [1, Theorem 12.3], we can suppose that $[M, x_{i_p}]$ converges to some point $[N, y]$ in $\mathcal{M}_*^\infty(n)$. Take a sequence of compact domains Ω_q in N such that $B_N(y, q+1) \subset \Omega_q$. For each q , there are pointed local embeddings $\phi_{q,p} : (N, y) \rightarrow (M, x_{i_p})$, for p large enough, such that $\Omega_q \subset \text{dom } \phi_{q,p}$ and $\phi_{q,p}^* g_M \rightarrow g_N$ on Ω_q with respect to the C^∞ topology. Let $h_{q,p} = \phi_{q,p}^* f$ on Ω_q . It is easy to see that, for all naturals q and m , the sequence $\|h_{q,p}\|_{C^m, \Omega_q, g_N}$ is uniformly bounded. Hence the functions $h_{q,p}$ form a compact subset of $C^\infty(\Omega_q, \mathbb{R}^{(c+1)(n+2)})$ with the C^∞ topology by [1, Proposition 3.11]. So some subsequence $h_{q,p(q,\ell)}$ is convergent to some $h_q \in C^\infty(\Omega_q, \mathbb{R}^{(c+1)(n+2)})$ with the C^∞ topology. In fact, arguing inductively on q , it is easy to see that we can assume that each $h_{q+1,p(q+1,\ell)}$ is a subsequence of $h_{q,p(q,\ell)}$, and therefore h_{q+1} extends h_q . Thus the functions h_q can be combined to define a function $h \in C^\infty(M, \mathbb{R}^{(c+1)(n+2)})$. Take sequences of integers, $\ell_q \uparrow \infty$ and $m_q \uparrow \infty$, so that

$$\|h - \phi_{q,p(q,\ell_q)}^* f\|_{C^{m_q}, \Omega_q, g_N} = \|h_q - h_{q,p(q,\ell_q)}\|_{C^{m_q}, \Omega_q, g_N} \rightarrow 0.$$

Then, considering h as an \mathbb{E} -valued function, we get that $[M, f, x_{i_p(q,\ell_q)}] \rightarrow [N, h, y]$ in $\widehat{\mathcal{M}}_*^\infty(n)$ as $q \rightarrow \infty$.

Now take an arbitrary sequence $[M, f, x'_{p_j}]$ in $\text{im } \hat{\iota}_{M,f}$. By (ii), there is a sequence of naturals, i_p , such that $d_M(x'_{p_j}, x_{i_p}) < r/2$. By the above case in the proof, after taking a subsequence if necessary, we can assume that $[M, f, x_{i_p}]$ is convergent to some point $[N, h, y]$ in $\widehat{\mathcal{M}}_*^\infty(n)$. Thus, given sequences, $m_j \uparrow \infty$ in \mathbb{N} , and $S_j \uparrow \infty$ and $s_j \downarrow 0$ in \mathbb{R}^+ , there is some sequence $p_j \uparrow \infty$ in \mathbb{N} such that there exists some $(m_j, S_j + e^{s_j} r/2, \lambda_j, \varepsilon_j)$ -pointed local quasi-equivalence $\phi_j : (N, h, y) \rightarrow (M, f, x_{i_{p_j}})$ for some $\lambda_j \in [1, e^{s_j})$ and $\varepsilon_j \in (0, s_j)$. Since $y'_j := \phi_j^{-1}(x'_{p_j}) \in B_N(y, e^{s_j} r/2)$, it follows that $\phi_j : (N, h, y'_j) \rightarrow (M, f, x'_{p_j})$ is an $(m_j, S_j, \lambda_j, \varepsilon_j)$ -pointed local quasi-equivalence, showing that $[M, f, x'_{p_j}] \in \widehat{U}_{S_j, s_j}^{m_j}(N, h, y'_j)$. On the other hand, since the sequence y'_j is bounded in N , we can suppose that it is convergent to some $y' \in N$ by taking a subsequence if necessary. Hence $[N, h, y'_j] \rightarrow [N, h, y']$ in $\widehat{\mathcal{M}}_*^\infty(n)$ by the continuity of $\hat{\iota}_{N,h}$. Hence there are sequences, $n_j \uparrow \infty$ in \mathbb{N} , and $T_j \uparrow \infty$ and $t_j \downarrow \infty$ in \mathbb{R}^+ , such that $[N, h, y'_j] \in \widehat{U}_{e^{s_j} T_j, t_j}^{n_j}(N, h, y')$ for j large enough. So

$$[M, f, x'_{p_j}] \in \widehat{U}_{S_j, s_j}^{m_j} \circ \widehat{U}_{e^{s_j} T_j, t_j}^{n_j}(N, h, y') \subset \widehat{U}_{\min\{S_j, T_j\}, s_j + t_j}^{\min\{m_j, n_j\}}(N, h, y')$$

for p large enough by Proposition 4.2-(iv). This shows that $[M, f, x'_{p_j}] \rightarrow [N, h, y']$ in $\widehat{\mathcal{M}}_*^\infty(n)$, completing the proof of Claim 6. \square

Proof of Theorem 1.1. Given a connected, complete Riemannian n -manifold M of bounded geometry, by Proposition 7.1, and Theorems 1.3 and 1.4, $\widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$ is a compact Riemannian foliated subspace of $\widehat{\mathcal{M}}_{*,\text{imm}}^\infty(n)$. Moreover $\hat{\iota}_{M,f} : M \rightarrow \text{im } \hat{\iota}_{M,f}$ is an isometry because f is an embedding. \square

8. OPEN PROBLEMS

Question 8.1. In Theorem 1.1, is it possible to get the Riemannian foliated space so that its leaves have trivial holonomy?

Question 8.1 can be reduced to the following question, in the same way as Theorem 1.1 follows from Proposition 7.1.

Question 8.2. In Proposition 7.1, is it possible to get f such that moreover¹¹ $\text{Iso}(N, h) = \{\text{id}_M\}$ if $\text{im } \hat{\iota}_{N,h} \subset \widehat{\text{Cl}}_\infty(\text{im } \hat{\iota}_{M,f})$?

In turn, Question 8.2 can be reduced to the following graph version. Consider only connected graphs with a countable set of vertices, all of them with finite degree. These graphs are proper path metric spaces in a canonical way so that each edge is of length one. Thus they define a subspace \mathcal{G}_* of the Gromov space \mathcal{M}_* of pointed proper metric spaces. Decorate such graphs with maps of their vertex set to \mathbb{N} . This gives rise to a space $\widehat{\mathcal{G}}_*$ of isomorphism classes of pointed decorated graphs, like in the case of $\widehat{\mathcal{M}}_*^\infty(n)$. Let $\widehat{\text{Cl}}$ denote the closure operator in $\widehat{\mathcal{G}}_*$. For each decorated graph (G, α) , let $\text{Iso}(G, \alpha)$ denote its group of isomorphisms. There is a canonical map $\hat{\iota}_{G,\alpha} : G \rightarrow \widehat{\mathcal{G}}_*$, like the above map $\hat{\iota}_{M,f}$. It is said that G is of *bounded geometry* if there is a uniform upper bound for the degree of its vertices.

Question 8.3. For any graph G of bounded geometry, does there exist a finite valued decoration α so that $\text{Iso}(H, \beta) = \{\text{id}\}$ for all decorated graph (H, β) with $\text{im } \hat{\iota}_{H,\beta} \subset \widehat{\text{Cl}}(\text{im } \hat{\iota}_{G,\alpha})$?

There are aperiodic tilings of \mathbb{R} (like the Fibonacci tiling), or elements of $\{0, 1\}^{\mathbb{Z}}$, giving rise to examples of decorations of the Cayley graph of \mathbb{Z} satisfying the condition of Question 8.3 (see e.g. [25]). If Question 8.3 had an affirmative answer, then, in the proof of Proposition 7.1, we could take a finite valued decoration α of G satisfying the condition of Question 8.3, and modify the definition of f so that

$$f^k(x) = (\tilde{\rho}^k(x) \cdot (\alpha(i) + 1/i), \tilde{\rho}^k(x) \cdot \tau \circ \kappa_i^{-1}(x))$$

if $x \in B_M(x_i, r)$ for some $i \in I^k$. This would give affirmative answers to Questions 8.2 and 8.1.

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¹¹According to the terminology for tilings, it could be said that (M, f) is *aperiodic* when this condition is satisfied. The same term could be also used for the corresponding property for graphs, in Question 8.3.

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